CS 341 — Discrete Math — Fall 1999

Problem Set 2 Answer Key

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1 Section 3.6, Ex. 6

We just partition the integers according to the number of digits they contain.
If the integer contains all 8 digits we can use, then the leftmost position has 7 choices(every digit except 0). So the number of such integers with 8 digits is $7 * 7 * 6 * 5 * 4 * 3 * 2 * 1$.
Similarly, the number of integers with 7 digits is $7 * 7 * 6 * 5 * 4 * 3 * 2$;
the number of integers with 6 digits is $7 * 7 * 6 * 5 * 4 * 3$;
the number of integers with 5 digits is $7 * 7 * 6 * 5 * 4$;
For integers with 4 digits, there are two cases:
if the leftmost digit is 5, then the number of integers is $4 * 6 * 5$;
if the leftmost digit is larger than 5, then the number of integers is $3*7*6*5$;
Add them together we get the final result: 94, 830

2 Section 3.6, Ex. 8

For the first question: the total number of different ways 15 people can be seated at a round table without any restrictions is 14! If B **MUST** sit next to A, then we can think of A and B as one person, and there are $2 * 13!$ ways to arrange them, where 2 stands for two different relative positions of B to A. We subtract the second number from the first one and get the result $14! - 2 * 13! = 12 * 13!$.
For the second question: similar reasoning, the result is $14! - 13! = 13 * 13!$.

3 Section 3.6, Ex. 10

There are totally $\binom{20}{3}$ sets of 3 numbers from 1 to 20.
Now we count the number of sets that contain at least two consecutive numbers:

If three numbers are all consecutive, then the number of sets is 18;
If only two numbers are consecutive, then there are two cases:
(a) If those two numbers are (1, 2) or (19, 20), then for each pair, there are 17 sets;
(b) If those two numbers are from (2, 3) to (18, 19), then for each pair, there are 16 sets;
So we know that there are $2 * 17 + 17 * 16$ number of sets in which only two numbers may be consecutive.
Then the final result should be $\binom{20}{3} - 18 - 16 * 17 - 17 * 2 = 816$.

4 Section 3.6, Ex. 23

Suppose there is a table, there are 4! ways for 5 men to seat around it. Then for each of those 4! ways, we let 5 women take seats such that every woman should take seat between two men, then there are 5! ways. *(Note that this number is 5! instead of 4!, because after seats of men have been fixed, it is no longer a circular permutation, but a normal one)* Then for each of those $4! * 5!$ ways to arrange seats for people, we insert dog between a man and a woman, there are 10 possible ways to do this. So the number of different ways to arrange men, women and a dog around a table should be $10 * 4! * 5!$ $= 2 * 5! * 5!$.

5 Section 3.6, Ex. 24

There are $15 * 14 * 13$ different ways to fix which three teams get awarded gold, silver and bronze cups repectively. Then after that, there are $\binom{12}{3}$ different ways to determine which three teams will be dropped to the lower league. So the number of different possible outcomes is: $\frac{15*14*13*12*11*10}{3!}$. *(Some students do not distinguish the first three teams, and so they got the answer* $\frac{15*14*13*12*11*10}{3!*3!}$*; Some others distinguish both the first three teams and the last three teams, and they got the answer* $15 * 14 * 13 * 12 * 11 * 10$. *Both interpretations are acceptable).*

6 Section 3.6, Ex. 28

all 3-combinations are: {a, a, b}, {a, a, c}, {a, b, c}, {a, c, c}, {b, c, c}, {c, c, c} all 4-combinations are: {a, a, b, c}, {a, a, c, c}, {a, c, c, c}, {a, b, c, c}, {b, c, c, c}

7 Special Job Offer Problem

7.1

There are 16 permutations which will result in a success, they are:

1 3 4 2	1 4 3 2	2 3 1 4	2 3 4 1
2 4 1 3	2 4 3 1	3 1 4 2	3 2 1 4
3 2 4 1	3 4 1 2	3 4 2 1	4 1 3 2
4 2 1 3	4 2 3 1	4 3 1 2	4 3 2 1

The other 8 permutations will result in a failure.
So $q_{4,2} = 16/24 = 2/3$

7.2

In class we have known that the number of permutations which will give us the best (number 1) offer is:

$$k(n-1)! \sum_{j=k+1}^{n} \frac{1}{j-1}$$

Now we consider the number of permutations which will give us precisely the number 2 job offer. Basically there are two different cases here:

(a) If job 1 occurs among the first k jobs, then in order to pick up job 2, it must be placed in the last position. So we have k possible positions for job 1(the first k positions), and (n-2)! possible permutations for the other n-2 jobs once we have fixed where to place job 1. So in this case, the number of possible permutations is $k(n-2)!$.

(b) If the job 1 does not occur among the first k jobs, then it must occur behind job 2, otherwise we won't have chance to pick up job 2. Meanwhile, job 2 can occur at any one position from k+1 to n-1. We use E_j to denote the set of all possible cases in which job 2 occurs at position j, where $k+1 \leq j \leq n-1$. Now let's calculate $|E_j|$ first. We have $\binom{n-2}{j-1}$ possible ways to choose j-1 jobs to be placed at position 1 to j-1. Moreover, once we have selected those j-1 jobs, the best one among them must be placed in one of the first k positions, otherwise we will pick up this job and cannot reach job 2. So there are k possible ways to place this "local best job". After that, the other j-2 jobs before job 2 can be arbitrarily arranged, so there are (j-2)! possible ways to arrange them. Now let's turn to those n-j jobs behind job 2, there are (n-j)! ways to arrange them. By Multiplication Principle, we know that $|E_j| = \binom{n-2}{j-1}k(j-2)!(n-j)!$, which is equal to $k(n-2)!\frac{n-j}{j-1}$. So totally the number of permutations in which job 1 does not occur among the first k jobs is:

$$k(n-2)!\sum_{j=k+1}^{n-1}\frac{n-j}{j-1}$$

Combining (a) and (b) and the former result given in class, we know that the final answer is:

$$q_{n,k} = k(n-1)!\sum_{j=k+1}^{n}\frac{1}{j-1} + k(n-2)! + k(n-2)!\sum_{j=k+1}^{n-1}\frac{n-j}{j-1}$$

(note: There are several other ways to do this problem, and it is possible that we derive different forms for the final result. For example, another possible answer is:

$$q_{n,k} = k(n-k+1)(n-2)! + 2k(n-2)!\sum_{j=k+1}^{n-1}\frac{n-j}{j-1}$$

It is not difficult to verify that those two different forms are equal.)