COS 341, November 10, 1999

Handout Number 7

Midterm Exam

Do all three problems. To get full credits, your answers must be given as closed-form expressions of n. In your derivations, you may use (no proof needed, just refer to the page number) any formula given in the textbook.

In the problems below, n is always a positive integer.

Problem 1 [20 pts] (a) Let $f(n) = \sum_{0 \le k \le n} (-1)^k k$. Prove that $f(n) = (-1)^n \lceil n/2 \rceil$.

(b) Let $g(n) = \displaystyle\sum_{k \ge 0} \frac{1}{4^k} \binom{n+k-1}{k}$. Determine $g(n)$.

Problem 2 [20 pts] Let n be any integer of the form m^2 for some positive integer m. Let S_n be the set of all pairs (i, j) such that (i) i, j are non-negative integers, and (ii) $i^2 + j \le n$. Let $h(n) = |S_n|$. Determine $h(n)$.

Problem 3 [20 pts] Alice and Bob each tosses an unbiased coin n times. Let X and Y be the random variables corresponding to the number of HEADs in Alice' and Bob's results.

(a) Let E_1 denote the event that $X = Y$, and let $s(n) = \Pr\{E_1\}$. Determine $s(n)$.

(b) Let E_2 denote the event that $X = Y + 1$, and let $t(n) = \Pr\{E_2\}$. Determine $t(n)$.

Remarks Note that $s(1) = 1/2$, $s(2) = 3/8$, $t(1) = 1/4$, $t(2) = 1/4$. You may want to check your answers for $n = 1, 2$ against these values.