COS 341, October 20, 1999
Handout Number 4

A Solution to Exercise 29 in 5.8

Let $1 \le k \le n$, and let $\mathcal{B}$ be a non-empty set of k-combinations of $\{1, 2, \cdots, n\}$. The *shadow* of $\mathcal{B}$ is defined as the set of all $(k-1)$-combinations T where $T \subseteq B$ for some $B \in \mathcal{B}$. For example, if $\mathcal{B} = \{\{1,2,3\},\{2,3,5\}\}$, then its shadow is $\{\{1,2\},\{1,3\},\{2,3\},\{2,5\},\{3,5\}\}$.

Lemma 1 If $\mathcal{B}$ is the set of *all* k-combinations, then $|\mathcal{T}|/|\mathcal{B}| = k/(n-k+1)$; otherwise $|\mathcal{T}|/|\mathcal{B}| > k/(n-k+1)$.

Proof Let G be the set of all pairs (B, T) such that $B \in \mathcal{B}$, T is a $(k-1)$-combination, and that $T \subseteq B$. Each B has exactly k T's associated with it in G, and thus

$$|G| = k \cdot |\mathcal{B}|. \tag{1}$$

Also, each T has *at most* $n-(k-1)$ B's associated with it. Thus,

$$|G| \le (n-(k-1)) \cdot |\mathcal{T}|. \tag{2}$$

Clearly, (1) and (2) imply

$$\mathcal{T}|/|\mathcal{B}| \ge k/(n-k+1). \tag{3}$$

To prove the lemma, note that when $\mathcal{B}$ is the set of all k-combinations, we have $\mathcal{T}|/|\mathcal{B}| = \binom{n}{k}/\binom{n}{k-1} = k/(n-k+1)$. We now prove the converse, ie, if (3) is an equality, then $\mathcal{B}$ must be the set of all k-combinations.

Assume equality holds in (3). Then equality in (2) must hold. That implies that, for every $T \in \mathcal{T}$, *all* the possible $n-(k-1)$ B's associated with T are in $\mathcal{B}$. That is, any $B \in \mathcal{B}$ implies that all B's *neighbors* must also be in $\mathcal{B}$, where a *neighbor* of B is any k-combination D such that $|D \cap B| = k-1$. But then $\mathcal{B}$ must contain all the k-combinations. (For any k-combination D, one can construct a sequence of k-combinations $D_0 = B, D_1, D_2, \cdots, D_m = D$ such that D_i and D_{i+1} are neighbors for all i. Since $D_0 = B$ is $\mathcal{B}$, we have by induction all D_i are in $\mathcal{B}$.)

This completes the proof of Lemma 1.$\Box$

For any clutter $\mathcal{Q}$, let $h_{\max}(\mathcal{Q})$ and $h_{\min}(\mathcal{Q})$ be, respectively, the largest and the smallest integer k for which there is a k-combination $B \in \mathcal{Q}$. Let $\overline{\mathcal{Q}} = \{\{1, 2, \cdots, n\} - B \mid B \in \mathcal{Q}\}$.

Lemma 2 If $\mathcal{Q}$ is a clutter, then $\overline{\mathcal{Q}}$ is also a clutter. Furthermore, $h_{\max}(\overline{\mathcal{Q}}) = n - h_{\min}(\mathcal{Q})$.

Proof Let $D, D' \in \overline{\mathcal{Q}}$ be any two distinct elements. By definition $D = \{1, 2, \cdots, n\} - B$, $D' = \{1, 2, \cdots, n\} - B'$ where $B, B' \in \mathcal{Q}$. Note that $D \subseteq D'$ if and only if $B \supseteq B'$. Since B and B' are by definition incomparable, we conclude that D and D' are also incomparable. This proves that $\overline{\mathcal{Q}}$ is a clutter. Now any $B \in \mathcal{Q}$ gives an element $D = \{1, 2, \cdots, n\} - B$ in $\overline{\mathcal{Q}}$, and vice versa. Thus, $n - h_{\min}(\mathcal{Q}) = h_{\max}(\overline{\mathcal{Q}})$. $\Box$

We now use Lemmas 1 and 2 to prove the result of Exercise 29. Note first that if $\mathcal{Q}$ is a counter-example to Exercise 29, then it must contain elements of different sizes; thus, for this $\mathcal{Q}$, $h_{\min}(\mathcal{Q}) \leq h_{\max}(\mathcal{Q}) - 1$, and by Lemma 2,

$$\begin{aligned} h_{\max}(\overline{\mathcal{Q}}) &= n - h_{\min}(\mathcal{Q}) \\ &\geq n + 1 - h_{\max}(\mathcal{Q}). \end{aligned}$$

It follows that for any counter-example $\mathcal{Q}$, we have

$$\max\{h_{\max}(\overline{\mathcal{Q}}), h_{\max}(\mathcal{Q})\} \geq (n+1)/2.$$

We note that $\overline{\mathcal{Q}}$ is also a counter-example.

To solve Exercise 29, we can thus, from the above discussions, assume that there is a counter-example clutter $\mathcal{Q}$ with $h_{\max}(\mathcal{Q}) \geq (n+1)/2$, and proceed to derive a contradiction.

Let $k = h_{\max}(\mathcal{Q}) \geq (n+1)/2$, and $\mathcal{B} \subseteq \mathcal{Q}$ be the set of all $B \in \mathcal{Q}$ with $|B| = k$. Let $\mathcal{T}$ be the shadow of $\mathcal{B}$. Define $\mathcal{V} = (\mathcal{Q} - \mathcal{B}) \cup \mathcal{T}$. Note that $(\mathcal{Q} - \mathcal{B}) \cap \mathcal{T} = \emptyset$ (*why?*), and thus $|\mathcal{V}| = |\mathcal{T}| + |\mathcal{Q} - \mathcal{B}|$. The next two lemmas show that $\mathcal{V}$ is a clutter with more elements than $\mathcal{Q}$, hence a contradiction.

Lemma 3 $\mathcal{V}$ is a clutter.

Proof Let D, D' be any two distinct elements in $\mathcal{V}$. We show that they are incomparable. If both D, D' are in $\mathcal{Q} - \mathcal{B}$, this is clearly true since $\mathcal{Q}$ is a clutter. If both are in $\mathcal{T}$, this is also true, since all elements of $\mathcal{T}$ are $(k-1)$-combinations (and thus incomparable). If $D \in \mathcal{Q} - \mathcal{B}$ and $D' \in \mathcal{T}$, then D cannot include D' since $|D| \leq k - 1$ and $|D'| = k - 1$; furthermore, D cannot be a subset of D' since D is not a subset of any element in $\mathcal{B}$ (and thus not a subset of any element in $\mathcal{T}$, the shadow of $\mathcal{B}$). The last case $D' \in \mathcal{Q} - \mathcal{B}$, $D \in \mathcal{T}$ can be treated similarly. This proves the lemma. $\Box$

Lemma 4 $|\mathcal{V}| > |\mathcal{Q}|$.

Proof Note that $|\mathcal{V}| = |\mathcal{T}| + |\mathcal{Q} - \mathcal{B}|$, and $|\mathcal{Q}| = |\mathcal{B}| + |\mathcal{Q} - \mathcal{B}|$. Thus, it suffices to prove $|\mathcal{T}| > |\mathcal{B}|$.

By the remarks after the proof of Lemma 2, $\mathcal{Q}$ is not the set of all the k-combinations. Thus, $\mathcal{B}$ cannot be the set of all the k-combinations, and by Lemma 1, we have

$$\begin{aligned} |\mathcal{T}|/|\mathcal{B}| &> k/(n-k+1) \\ &= (n+1)/(n-k+1)-1 \\ &\geq (n+1)/(n-(n+1)/2+1)-1 \\ &= 1. \end{aligned}$$

This proves $|\mathcal{T}| > |\mathcal{B}|$, and hence Lemma 4. $\square$