COS 341, December 1, 1999
Handout Number 10

Sample Exam

This is a copy of the take-home final exam given last year.

Problem 1 [30 points] For each of the following statements, give a "True" or "False" answer. Justify your answers concisely.

(a) [10 points] Let n be any positive integer. Any graph G on n vertices must satisfy either $\omega(G) \geq (\log_4 n) - 1$ or $\alpha(G) \geq (\log_4 n) - 1$. (Recall that $\omega(G)$ is the maximum size of any clique of G, and that $\alpha(G)$ is the maximum size of any independent set of G.)

(b) [10 points] Let n be any positive integer. Let G be a graph on n vertices satisfying $\deg(v) \geq n/2$ for all vertices v. Then between any two distinct vertices u and v, there exists in G a path of length at most $n/2$ that connects u and v.

(c) [10 points] Let n be any positive integer. Let $G = (V, E)$, where $V = \{1, 2, \cdots, n\}$ and E is the set of all $\{i, j\}$ $(1 \leq i < j \leq n)$ satisfying $0 < |i - j| \leq 3$. Then G is a planar graph.

Problem 2 [25 points] Let $n = 6s$ where s is a positive integer. Let a_n be the number of triplets of integers (i, j, k) such that (1) i, j, k are non-negative and distinct integers (ie, $i \geq 0, j \geq 0, k \geq 0, i \neq j, j \neq k, i \neq k$), and (2) $i + j + k \leq n$. Derive an explicit closed-form expression for a_n.

Problem 3 [25 points] Let $n \geq m > 0$ be integers. A binary string $u = u_1 u_2 \cdots u_n$ is said to *contain* $v = v_1 v_2 \cdots v_m$ *as a substring* if there exists an integer i (where $0 < i \leq n - m + 1$) such that $u_i = v_1, u_{i+1} = v_2, \cdots, u_{i+m-1} = v_m$. For example, 110100 contains 101 as a substring (with $i = 2$), while the string 001100 does not contain 101 as a substring.

Let b_n denote the number of binary strings of length n that contain 101 as a substring. Clearly, $b_1 = b_2 = 0$. Let $B(x) = \sum_{n \geq 1} b_n x^n$.

(1) [5 points] Determine the value of b_5.

(2) [20 points] Derive an explicit closed-form expression for $B(x)$.

Hint You might want to first set up recurrence relations for the appropriate sequences.

Remarks For $n = 3$, there is only one binary string of length n that contains 101 as a substring (ie, 101). For $n = 4$, there are exactly four binary strings of length n that contain 101 as a substring (ie, $0101, 1010, 1011, 1101$). Thus, $b_3 = 1, b_4 = 4$.