Lecture Notes #10 - Surfaces

- Reading:
  - Angel: Chapter 9
  - Foley: Chapter 11 (optional)

- Sweep Surfaces
  - Surfaces of revolution
  - General sweeps
- Tensor Product Surfaces
- Subdivision Surfaces

Motivation

Q: Why "do" surfaces at all?

- Animation and movies
- Aerodynamic, stress, fluid-flow simulations
- Numerically-controlled manufacturing
- Virtual reality

Surfaces of revolution

Idea: Rotate a 2D "profile" curve around an axis.

Q: What kinds of shapes can you model this way?

Background: 2D rotation

Given: A point $P = (p_x, p_y)$ and an angle θ.

Find: Coordinates (p_x', p_y') of P rotated by θ.

Solution:

1. Write P' in $(\hat{x}', \hat{y}')$ coordinate system:

2. Write $\hat{x}'$ and $\hat{y}'$ in terms of $\hat{x}$ and $\hat{y}$:

Background: 2D rotation, cont.

The result P' can also be expressed in matrix notation:

$$P' = \begin{pmatrix} \hat{x}' & \hat{y}' & O' \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{x} & \hat{y} & O \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

Note: $R(\theta)$ is good for rotating any point:

$$\begin{pmatrix} p_x' \\ p_y' \\ 1 \end{pmatrix} = R(\theta) \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

Background: 3D rotation

Given: A point $P = (p_x, p_y, p_z)$ and an angle θ.

Find: A rotation matrix $R_z(\theta)$ such that

$$P' = R_z(\theta)\, P$$

Solution: (same ideas as before):

- Write $\hat{x}'$, $\hat{y}'$, $\hat{z}'$ in terms of $\hat{x}$, $\hat{y}$, $\hat{z}$:

Background: 3D rotation, cont.

Gives:

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise: Find $R_x(\theta)$, $R_y(\theta)$

Constructing a surface of revolution

Given: A curve $C(u) : u \in [0, 1]$ in the yz-plane:

$$C(u) = \begin{pmatrix} 0 \\ c_y(u) \\ c_z(u) \\ 1 \end{pmatrix}$$

Find: A survace $S(u, v) : u, v \in [0, 1]$, which is $C(u)$ rotated about the z-axis.

Solution:

Q: How do you display such a surface?

General sweep surfaces

The "surface of revolution" is a special case of a "sweep surface" ...

Idea: Trace out surface $S(u, v)$ by moving a "profile" curve $C(u)$ along a "trajectory" $T(v)$.

More specifically:

- Suppose that $C(u)$ lies in an $(\hat{x}_c, \hat{y}_c)$ coordinate system with origin O_c.
- For every point along $T(v)$, lay in $C(u)$ so that $T(v)$ coincides with O_c.

Orientation

The big issue:

- How to orient $C(u)$ as it moves along $T(v)$.

Lots of options: Here are two:

1. Fixed (or "static"): Just translate O_c along $T(v)$:

2. Moving: Use the "Frenet frame" of $T(v)$.
   - Allows smoothly varying orientation
   - Permits surfaces of revolution, for example

Frenet frames

Motivation: Given a curve $T(v)$, we want to attach a smoothly-varying coordinate system.

Idea: To get a 3D coordinate system, we need 3 independent direction vectors.

- Tangent $\hat{t}(v) = normalize(T'(v))$
- Bi-normal $\hat{b}(v) = normalize(T'(v) \times T''(v))$
- Normal $\hat{n}(v) = normalize(\hat{b}(v) \times \hat{t}(v))$

As we move along $T(v)$, the Frenet frame $(\hat{t}, \hat{b}, \hat{n})$ varies smoothly.

Frenet swept surfaces

Idea: Orient the profile curve $C(u)$ using the Frenet frame of the trajectory $T(v)$:

- Put $C(u)$ in the "normal" plane $\widehat{nb}$.
- Place O_c on $T(v)$.
- Align $\hat{x}_c$ for $C(u)$ with $-\hat{n}$.
- Align $\hat{y}_c$ for $C(u)$ with $\hat{b}$.

Cool: If $T(v)$ is a circle, you get a surface of revolution exactly!

Problem: What do you do when the curvature changes sign or goes to 0?

Variations

Several variations are possible:

- Scale $C(u)$ as it moves, possible using length of $T(v)$ as a scale factor -- great for seashells!
- Morph $C(u)$ into some other curve $C'(u)$ as it moves along $T(v)$.
- Use your imagination....

Bezier tensor product surfaces

Given: A grid of control points V_{ij} : $i, j = 0, \ldots, n$.

Construct: A surface $S(u, v)$ by

- Treating rows of V as control points for curves $C_0(u), \ldots, C_n(u)$.
- Treating $C_0(u), \ldots, C_n(u)$ as control points for a curve parameterized by v.

Q: Which control points are interpolated by the surface?

Matrix form

Tensor product surfaces can be written out explicitly:

$$S(u,v) = \sum_{i,j=0,\ldots,n} V_{i,j}\, B_i^n(u)\, B_j^n(v)$$

$$= \begin{pmatrix} v^3 & v^2 & v & 1 \end{pmatrix} M_{Bez}\, V\, M_{Bez} \begin{pmatrix} u^3 \\ u^2 \\ u \\ 1 \end{pmatrix}$$

Tensor Product B-splines

- Defined by an array of control points $\{P_{0,0},\ldots,P_{m,m'}\}$
- Composed of an array of surface patches. The k,l-th patch can be written as

$$S(u,v) = \sum_{i,j=-1,\ldots,2} P_{k+i,j+l}\, N_i(u) N_j(v)$$

$$= \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} M_{Bspline}\, P\, {M_{Bspline}}^T \begin{pmatrix} v^3 \\ v^2 \\ v \\ 1 \end{pmatrix}$$

Topological Type

- So far: can only model surfaces of restricted topological type:
  - Topological disks
  - Topological cylinder
  - Topological sphere (genus 0) -- degenerate parametrization
  - Topological torus (genus 1)
- But not
  - Surfaces of genus $g > 1$

Subdivision Surfaces

- Capable of modeling arbitrary topological types
- Idea:
  - Start with arbitrary triangulation
  - Repeatedly refine by:
    - Splitting each face into 4 faces
    - Average