Solutions to Homework Set 8

Section 12.6, Problem 5

Call the three given graphs G_1, G_2 and G_3 from left to right. We see that G_1 contrains K_2 as a subgraph, G_2 contains K_3 and G_3 contains K_4. This means $\chi(G_1) \geq 2, \chi(G_2) \geq 3, \chi(G_3) \geq 4$. By a little trial-and-error it is easy to come up with a 2-coloring of G_1, a 3-coloring of G_2 and a 4-coloring of G_3. Thus, $\chi(G_1) = 2, \chi(G_2) = 3$ and $\chi(G_3) = 4$. $\square$

Section 12.6, Problem 18

The simplest example of a planar graph with chromatic number 4 that doesn't contain a K_4 is the *5-wheel*. For integers $n \geq 3$, an *n-wheel* is defined as an n-cycle with one special vertex added; the special vertex is adjacent to all n vertices forming the cycle.

An n-wheel is clearly planar, 4-colorable and, if $n \geq 4$, does not contain a K_4. Why is a 5-wheel not 3-colorable? Since the "rim" of the wheel (the 5-cycle) is an *odd cycle* it requires at least three colors. The color of the special (the 6th) vertex must be different from that of any of the vertices on the cycle. Hence the result. $\square$

Section 12.6, Problem 19

Proof by induction: If we have just one line in the plane then we can clearly do the required 2-coloring by just coloring the two regions thus created differently. Now suppose the set of regions of a plane created by n dividing lines is 2-colorable. Consider adding an $(n+1)^{st}$ line to a properly colored plane divided by n lines. The new line splits the entire plane into two half-planes, each of which is correctly colored. Flip the color of every region in one of the half-planes. This still keeps each half-plane correctly colored. But now if two regions share the $(n+1)^{st}$ line for a boundary, they are differently colored, because

- before adding the $(n+1)^{st}$ line they must have been one region
- thus, before the color flipping they must have been colored the same.

Thus, after the flipping, the entire plane is correctly colored. $\square$

Direct proof: This proof relies less on geometric intuition but one needs to work a little to make it rigorous. Consider a graph whose vertex set is the set of regions formed by the n lines and where two vertices are adjacent iff the corresponding regions share a boundary in the plane. We must show that this graph is 2-colorable, i.e., bipartite, i.e., that every cycle in it is of even length.

Each of the n-lines divides the plane into two half-planes; arbitrarily associate the bit "0" with one of them (half-planes) and the bit "1" with the other. Now every region in the plane (every vertex in the graph) can be associated with an n-bit string where the i^{th} bit says whether the region lies in the "0-half-plane" or the "1-half-plane" associated with the i^{th} line. Identify each vertex with its associated bit string.

Two adjacent bit strings (vertices) clearly differ in exactly one bit position. But if we have a cycle of length k in the graph, then traversing around the cycle we should see exactly k bit-flips. However, we must end up with the same bit string we started with; hence each bit must get flipped an even number (possibly 0) of times. Hence k must be even. $\square$

Section 12.6, Problem 30

A tree contains no cycles; in particular, no cycles of odd length. Therfore *a tree is bipartite*. Let T be a tree, $\chi(T) \leq 2$. But $\chi(T) \geq n/\alpha(T)$. Therefore $\alpha(T) \geq \lceil n/\chi(T) \rceil \geq \lceil n/2 \rceil$. $\square$

Note that a null graph is also a tree, so the chromatic number of a tree could be 1. It is a subtle point, but it cannot be neglected.

Section 12.6, Problem 42

An induced sugraph H of a bipartite graph G is bipartite. Therefore, H has no odd cycles; in particular, no triangles. Therefore $\omega(H) \leq 2$. By bipartiteness, $\chi(H) \leq 2$. Now suppose $\omega(H) = 1$. Then H must be a null graph and so $\chi(H) = 1$. If $\omega(H) = 2$ then, since $\chi(H) \geq \omega(H)$ and $\chi(H) \leq 2$, we get $\chi(H) = 2$. Thus, in any case, $\omega(H) = \chi(H)$.

Since H was an *arbitrary* induced subgraph, this is true of *all* induced subgraphs of G. Therefore, G is χ-perfect; hence perfect. □

Special Problem 1

(a) $t3 = 15; t4 = 56$.
(b) Everyone got the recurrrence relation correct.
$t_n = 3 * t_{n-1} + s_{n-1}$
$s_n = 2 * t_{n-1} + s_{n-1}$
So $T(x) = \sum_{n\geq 1} t_n x^n$
$S(x) = \sum_{n\geq 1} s_n x^n$
$T(x) = \sum_{n\geq 2} (3t_{n-1} + s_{n-1})x^n + t_1 x = 3xT(x) + xS(x) + x;$
$S(x) = s_1 x + \sum_{n\geq 2} 2t_{n-1} + s_{n-1} x^n = x + 2xT(x) + xS(x)$
$S(x) = \frac{x}{1-x} + \frac{2x}{1-x} T(x)$
Then $T(x) = \frac{x}{x^2-4x+1}$ □

Special Problem 2

(a) $\omega(G) = 5$. The clinique is made of v_1, v_2, v_3, v_4, v_5.
(b)$\alpha(G) = 4$. The independent set can be w_1, w_2, w_3 and v_4 or v_5.
(c)$\chi(G) = 5$. Since G contains aK_5, so $\chi(G) \geq 5$. And we can color w_i with the same color as v_4 or v_5.
(d)G is NOT a planar graph, since it contains a K_5.
(e)G is not an Eulerian graph, since 3 vertices of G has odd degree.
(f)G doesn't have a Hamiltonian cycle, since G has a subgraph which is a complete bipartite graph $K_{3,3}$. □

Comments? Corrections? Suggestions? Mail to `qiexh@cs.princeton.edu`