Solutions to Homework Set 7

Section 11.9, Problem 5

In a graph of order n, each vertex can have have degree of 0 through n-1. However, 0 and n-1 are mutually exclusive values. Because if there is a vertex with degree of n-1, then it is connected to every other vertices in this graph, thus no vertex can have degree of 0, and vice versa. So there are only n-1 choices of degree for n vertices. By pigeon-hole principle, at lease two vertices will have the same degree. □

This does not hold true for multigraph. Since each vertex can have as much edges as it likes.

Section 11.9, Problem 12

The leftmost graph is isomorphic to the middle graph and the only way to show this is by explicitly giving labelings for the two graphs such that (i, j) is an edge in the first iff (i, j) is an edge in the second. Fortunately, many such labelings exist and it is easy to find one.

These two graphs have no 4-cycles. However the rightmost graph has one (well, at least one)! Therefore it cannot be isomorphic to either of the first two graphs. □

Section 11.9, Problem 20

Once again, lots of bogus arguments were submitted for this problem. Following is a correct one:

For a disconnected graph, we can divide it into two parts with k vertices in one and $n - k$ in the other $(1 \leq k \leq n - 1)$. Note there are no edges connecting the two parts. So the maximun edges we can have in a disconnected graph of n vertices is:

$$\binom{k}{2} + \binom{n-k}{2}$$

$$= \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2}$$

$$\leq \frac{(n-1)(k-1)}{2} + \frac{(n-1)(n-k-1)}{2}$$

$$= \frac{(n-1)(n-2)}{2}$$

Hence $\frac{(n-1)(n-2)}{2} + 1$ edges cannot be accomodated in a disconnnected graph.

As for the example of an n-vertex graph with $(n - 1)(n - 2)/2$ edges that is not connected, simply consider K_{n-1} with one isolated vertex added. **Important:** *it is not enough to give an example for, say,* $n = 5$.

Section 11.9, Problem 26

This problem is too easy. Almost everyone got it right.

Section 11.9, Prolbem 29

Everyone got the correct answer also.

Section 11.9, Problem 40

We will prove that G satisfies the Ore property.

If G is complete, then it clearly has a Hamilton cycle, so we need not consider this case anymore. So let u, v be two non-adjacent vertices of G. There can be at most $\binom{n-2}{2} = (n - 2)(n - 3)/2$

edges amongst the vertices of G other than u, v. Each of the remaining edges is incident to either u or v, so it contributes 1 to the sum $\deg u + \deg v$. Thus

$$\deg u + \deg v \geq \frac{(n-1)(n-2)}{2} + 2 - \frac{(n-2)(n-3)}{2} = n.$$

Since u and v were *any two* non-adjacent vertices, we have verified the Ore property. $\square$As for the example of an n-vertex graph with $(n-1)(n-2)/2+1$ edges that does not have a Hamilton cycle, simply consider K_{n-1} with one extra vertex joined to it by a single edge. Since this "single edge" is a bridge in the graph, the graph cannot have a Hamilton cycle.

Special problem

This was a completely mechanical problem, so I've given credit for wrong final answers which were wrong only because of some "careless copying errors". The graph G_S which you were asked to construct has 12 vertices. After some trial and error you'll see that it has exactly *three* Eulerian trails, all starting at vertex GAC and ending at CGT. Following the Eulerian trails gives you three possible $\overline{\sigma}$'s, hence three possible σ's:

$$CTGCTTGACTGAGTGCA$$

$$CTGACTCGTTGAGTGCA$$

$$CTGAGTGCTTGACTGCA$$

and it's now easy to infer that the dinosaur is vegetarian and can fly but may not be timid.

The edges of G_S can be partitioned into either three or two edge-disjoint cycles. The former partitioning gives a G'_S which is a triangle while the latter gives a G'_S which has two vertices connected by three parallel edges. $\square$

Comments? Corrections? Suggestions? Mail to `qiexh@cs.princeton.edu`