Solutions to Homework Set 6

Section 7.8, Problem 7

Suppose $n > 2$. If we color the first cell white or blue then we may color the rest of the board in h_{n-1} ways in each case. If, instead, we color it red then we are forced to color the second cell either white or blue. In each of these subcases we may color the remaining $n-2$ cells in h_{n-2} ways. Thus

$$h_n = 2h_{n-1} + 2h_{n-2}.$$

We also note that $h_1 = 3$ and $h_2 = 8$ by simply counting all colorings.

This is a linear homogeneous recurrence and thus we solve it using the standard procedure described in the text. The characteristic equation $x^2 - 2x - 2 = 0$ has roots $1 \pm \sqrt{3}$. Thus the general solution is

$$h_n = c_1(1+\sqrt{3})^n + c_2(1-\sqrt{3})^n.$$

Using the initial values $h_1 = 3$ and $h_2 = 8$ we can solve for c_1 and c_2. Doing so gives

$$c_1 = \frac{\sqrt{3}+2}{2\sqrt{3}}, \quad c_2 = \frac{\sqrt{3}-2}{2\sqrt{3}}.$$

□

Section 7.8, Problem 15

(a) $h_n = 3^n$

(e) $h_n = 2^{n+1} - 1$

Every one got this right. □

Section 7.8, Problem 17

We distinguish one point p on the circle. p can pair with any other points as long as the number of points on either side of the line segment containing p is even. Suppose the line segment containing p divides the circle into two parts with $2k$ and $2n-2k-2$ points respectively, then there are $h_k h_{n-k-1}$ ways to join the points without intersecing for this particular choice. Hence, sum up all possible k,

$$h_n = \sum_{k=0}^{n-1} h_k h_{n-k-1}$$

□

Section 7.8, Problem 33

From Newton's binominal theorem we know

$$\frac{1}{(1-x)^4} = \sum_{k=0}^{\infty} \binom{4+k-1}{k} x^k = \sum_{k=0}^{\infty} \binom{k+3}{3} x^k$$

So

$$\sum_{n=0}^{\infty} \binom{n}{3} x^n = \sum_{n=3}^{\infty} \binom{n}{3} x^n = \sum_{k=0}^{\infty} \binom{k+3}{3} x^{k+3}$$

$$= x^3 \sum_{k=0}^{\infty} \binom{k+3}{3} x^k = \frac{x^3}{(1-x)^4}$$

That's the generating function for the sequence $h_n = \binom{n}{3}$. □

Special Problem 1

For the convex polygon with $n+1$ sides, the number of regions is h_{n-1}. Now and a new point A and still make the polgon be convex. Suppose A happens to insert between vertex B and C. The triangle ABC is divided into n regions obviously. And The original h_{n-1} regions are divided to more parts. The number of extra regions equals to the numbers intersections between the diagonals starting from A and the original diagonals inside the $n+1$-side polygon. We can count the intersections by sweeping from left to right. So the number of intersections equals to:
$1*(n-1)+2*(n-2)+...+(n-1)*1=\sum_{k=1}^{k<n} k*(n-k)=\binom{n+1}{3}$.
So $h_n = h_{n-1}+\binom{n+1}{3}+n$ is proven.

The generating function is:
$A(x)=\sum_{n\geq 0} h_n x^n = \sum_{n\geq 1} h_n x^n$
$=\sum_{n\geq 1} h_{n-1}x^n + \sum_{n\geq 1}\binom{n+1}{3}x^n + \sum_{n\geq 1} nx^n$
$= x*A(x)+\frac{1}{x}\sum_{n\geq 2}\binom{n}{3}x^n + \sum_{n\geq 1}\binom{n}{1}x^n$

Since $\sum_{n\geq 2}\binom{n}{3}x^n = \frac{x^3}{(1-x)^4}, \sum_{n\geq 1}\binom{n}{1}x^n = \frac{x}{(1-x)^2}$
$A(x) = x*A(x)+\frac{x^2}{(1-x)^4}+\frac{x}{(1-x)^2}$
$A(x)=\frac{x^2}{(1-x)^5}+\frac{x}{(1-x)^3}$
Also, $\frac{1}{(1-x)^5}=\sum_{n=0}^{\infty}\binom{n+4}{n}x^n$, $\frac{1}{(1-x)^3}=\sum_{n=0}^{\infty}\binom{n+2}{n}x^n$
Then $A(x)=x^2\sum_{n=0}^{\infty}\binom{n+4}{n}x^n + x\sum_{n=0}^{\infty}\binom{n+2}{n}x^n$
$= x+\sum_{n\geq 2}\binom{n+2}{4}x^n+\sum_{n\geq 2}\binom{n+1}{2}x^n$.
so $h_n=\binom{n+2}{4}+\binom{n+1}{2}$ for $n\geq 2$. □

Special Problem 2

There are many ways to solve this problem and many correct solutions were received too. At the outset we observe that a we may never move west if we are following a legal path. METHOD ONE: At each N-S street except the last you have to make a decision: to continue on the E-W street you are currently on or to switch to the other E-W street. To choose a path from Q to W we thus have to make $n-1$ decisions with 2 choices each time. Thus, we may choose a path in 2^{n-1} ways and so there are 2^{n-1} paths from Q to W.

METHOD TWO: A path from Q to W is uniquely defined by the set of N-S streets it traverses. This set must be of odd size since we eventually must end up at a different E-W street from the one we started on. Thus, the number of paths equals the number of odd-sized subsets of an n-element set which, of course, equals 2^{n-1}.

METHOD THREE: Let R be the point directly north of Q. Let $h(n)$ denote the number of ways to get from R to W without traversing a segment twice. In a path from Q to W we may start either by moving north, then east, and completing the path in one of $h(n-1)$ ways or by moving east and completing the path in one of $g(n-1)$ ways. Thus

$$g(n)=g(n-1)+h(n-1), \quad \text{for } n>2.$$

A similar argument shows that

$$h(n)=g(n-1)+h(n-1), \quad \text{for } n>2.$$

Thus $g(n)=h(n)$ for $n>2$ and we easily see that this holds for $n=2$ as well. Therefore, $g(n)=2g(n-1)$ and $g(2)=2$, whence $g(n)=2^{n-1}$.

□

Special Problem 3

The given recurrence is not linear homogeneous and so we cannot apply our standard method directly. However, if we define $c_n = 1/d_n$ we have

$$c_n = \frac{6d_{n-1} + d_{n-2}}{d_{n-1}d_{n-2}} = c_{n-1} + 6c_{n-2}.$$

Now this *is* a a linear homogeneous recurrence, so we *can* apply our standard method. The characteristic equation $x^2 - x - 6 = 0$ has roots -2 and 3, whence the general solution is: $c_n = a_1(-2)^n + a_2 3^n$. From the initial values $d_0 = d_1 = 1$ we get $c_0 = c_1 = 1$. Plugging these in the general solution and solving for a_1, a_2 gives

$$a_1 = 2/5. \;\; a_2 = 3/5.$$

And now,

$$c_n = \frac{2}{5}(-2)^n + \frac{3}{5}3^n = \frac{3^{n+1} - (-2)^{n+1}}{5}$$
$$d_n = \frac{1}{c_n} = \frac{5}{3^{n+1} - (-2)^{n+1}}.$$

□

Comments? Corrections? Suggestions? Mail to `qiexh@cs.princeton.edu`