Solutions to Homework Set 4

Problem 5:

Let $a = 2x, \quad b = -y$. We are then looking to expand $(a+b)^7$, which is simply:

$$\binom{7}{0}a^0b^7 + \binom{7}{1}a^1b^6 + \binom{7}{2}a^2b^5 + \binom{7}{3}a^3b^4 + \binom{7}{4}a^4b^3 + \binom{7}{5}a^5b^2 + \binom{7}{6}a^6b^1 + \binom{7}{7}a^7b^0$$

$$= b^7 + 7ab^6 + 21a^2b^5 + 35a^3b^4 + 35a^4b^3 + 21a^5b^2 + 7a^6b + a^7$$

And by substituting a=2x, b=-y, we obtain:

$$= (-y)^7 + 7(2x)(-y)^6 + 21(2x)^2(-y)^5 + 35(2x)^3(-y)^4 + 35(2x)^4(-y)^3 + 21(2x)^5(-y)^2 + 7(2x)^6(-y) + (2x)^7$$

$$= -y^7 + 14xy^6 - 84x^2y^5 + 280x^3y^4 - 560x^4y^3 + 672x^5y^2 - 448x^6y + 128x^7$$

Problem 10:

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{n-1} + \binom{n-3}{k-1}$$

First, let us assume a set S with three distinct elements, a, b, and c, and with a total of n elements ($n \geq 3$).

On the left-hand side of the equal sign, let $\binom{n}{k}$ be the number of ways to select k-combinations of S. Let $\binom{n-3}{k}$ be the number of k-combinations of S not containing a, b, or c. Note that $\binom{n}{k} - \binom{n-3}{k}$ is thus the number of k-combinations of S containing at least one of a, b, or c.

On the other hand, let $\binom{n-1}{k-1}$ be the number of ways to select k-combinations of S containing a. Let $\binom{n-2}{k-1}$ be the number of ways to select k-combinations of S containing b but not a (so as to not overcount those combinations with both a and b). Let $\binom{n-3}{k-1}$ be the number of ways to select k-combinations of S containing c but not a or b (again to not overcount). Note, then, that $\binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$ is the number of ways to select k-combinations of S containing a or b or c. And thus, both sides are combinatorially equivalent.

Problem 19:

There are clearly three basic cases with which we must be concerned: where $k \leq -1$, $k = 0$, and $k \geq 1$.

If $k \leq -1$, $\binom{-r}{k}$ is by definition 0. $\binom{r+k-1}{k}$ is also by definition 0, since in both cases $k < 0$. Thus, in this case the equation is satisfied.

If $k = 0$, $\binom{-r}{k}$ is by definition 1. $\binom{r+k-1}{k}$ is also by definition 1, since in both cases $k = 0$. Moreover, $(-1)^k = (-1)^0 = 1 \Rightarrow (-1)^k\binom{r+k-1}{k} = 1$, and the equation is once again satisfied.

In the final case, $k \geq 1$, we note that

$$\binom{-r}{k} = \frac{(-r)(-r-1)(-r-2)\ldots(-r-k+1)}{k!}$$

And by pulling out -1 from each of the k terms on top, we get

$$\binom{-r}{k} = (-1)^k\frac{(r)(r+1)(r+2)\ldots(r+k-1)}{k!} = (-1)^k\frac{(r+k-1)!}{k!(r-1)!}$$

But we note that this is simply

$$\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}$$

And again the equation is satisfied.

Problem 29:

Note that this problem is simply the question of maximizing the size of a clutter. The talk show host has a set of 10 jokes, and you want to select as many combinations of them as possible (each combination is the set of jokes he tells each night) without having any subsets (all jokes told on one night are told on another night). Thus, the maximum number of nights you can tune in is $\binom{10}{5} = 252$. This follows from Sperner's Theorem.

Problem 30:

From binomial theorem we know the coefficient of x^n in $(1+x)^{m_1+m_2}$ is $\binom{m_1+m_2}{n}$. If we look it as a product of $(1+x)^{m_1}$ and $(1+x)^{m_2}$, we can produce the term x^n by choosing k x's from $(1+x)^{m_1}$ (there are $\binom{m_1}{k}$ choices) and then $n-k$ x's from $(1+x)^{m_2}$ (there are $\binom{m_2}{n-k}$ choices). Hence for a specific k, by multiplication principle there are $\binom{m_1}{k} \times \binom{m_2}{n-k}$ choices to produce x^n. Sum all possible k, which is from 0 to n, we'll have the coefficient of x^n in $(1+x)^{m_1+m_2}$ also equals $\sum_{k=0}^{n} \binom{m_1}{k} \times \binom{m_2}{n-k}$. Thus the two expressions are identical.

Special Problem 1:

Let us start out by noting that

$$\sum_{0 \le i \le m} x^i = \frac{1-x^{m+1}}{1-x}$$

Taking the derivative of both sides yields:

$$\sum_{0 \le i \le m} i x^{i-1} = \frac{mx^{m+1} - (m+1)x^m + 1}{(1-x)^2}$$

And, multiplying both sides by x:

$$\sum_{0 \le i \le m} i x^i = \frac{mx^{m+2} - (m+1)x^{m+1} + x}{(1-x)^2}$$

In our specific case, $x = 2, \quad m = n-1$ yields the following two equations of relevance:

$$\sum_{0 \le k \le n-1} 2^k = \frac{1-2^n}{1-2} = 2^n - 1$$

$$\sum_{0 \le k \le n-1} k 2^k = \frac{(n-1)2^{n+1} - n2^n + 2}{(1-2)^2} = (n-1)2^{n+1} - n2^n + 2$$

Finally, we look at the given problem:

$$\sum_{0 \le k \le n-1} 2^k \frac{\binom{n-1}{k}}{\binom{n}{k}} = \sum_{0 \le k \le n-1} 2^k \frac{\frac{(n-1)!}{k!(n-k-1)!}}{\frac{n!}{k!(n-k)!}}$$

$$= \sum_{0 \le k \le n-1} 2^k \frac{n-k}{n}$$

$$= \sum_{0 \le k \le n-1} 2^k - \frac{1}{n} \sum_{0 \le k \le n-1} k2^k$$

$$= 2^n - 1 - \frac{1}{n}((n-1)2^{n+1} - n2^n + 2)$$

$$= 2^n - 1 - 2^{n+1} + \frac{2^{n+1}}{n} + 2^n - \frac{2}{n}$$

$$= 2^{n+1} - 2^{n+1} - 1 + \frac{2^{n+1}}{n} - \frac{2}{n}$$

$$= \frac{2^{n+1}}{n} - \frac{2}{n} - 1$$

$$= \frac{2^{n+1} - 2 - n}{n}$$

Special Problem 2:

Let $N = 10^4$, $1 \le n \le N$. If $\lfloor n^{1/4} \rfloor$ divides n, then you win \$5. Otherwise, you lose \$1. Let us start by counting the total number of ways one could win. In order to win, $\lfloor n^{1/4} \rfloor$ must divide n. There are 10 possibilities for the value of $\lfloor n^{1/4} \rfloor$ $(1, 2, \ldots, 9, 10)$, each with a corresponding range of n's $(1^4 \ldots 2^4 - 1, 2^4 \ldots 3^4 - 1, \ldots, 9^4 \ldots 10^4 - 1, 10^4)$

Every jth integer in the range $j^4 \le n \le (j+1)^4 - 1$ satisfies j dividing n. In each of these ranges, there are $\lceil \frac{(j+1)^4 - 1 - j^4 + 1}{j} \rceil$ divisible integers.

$$\lceil \frac{(j+1)^4 - 1 - j^4 + 1}{j} \rceil = \lceil \frac{j^4 + 4j^3 + 6j^2 + 4j + 1 - 1 - j^4 + 1}{j} \rceil$$

$$= \lceil 4j^2 + 6j + 4 + \frac{1}{j} \rceil$$

$$= 4j^2 + 6j + 5$$

Thus, the total number of winning spots on the roulette wheel is:

$$N_{win} = 1 + \sum_{j=1}^{9} (4j^2 + 6j + 5)$$

where the 1 comes from the possibility of hitting 10^4 precisely, which is not included in the sum because it is the only winning value for its range.

Using $j^2 = 2\binom{j}{2} + j$ and $\sum_{j=1}^{k} \binom{j}{2} = \binom{k+1}{3}$, reduce the above sum to:

$$1 + \sum_{j=1}^{9} (8\binom{j}{2} + 4j + 6j + 5)$$

$$= 1 + 8\sum_{j=1}^{9} \binom{j}{2} + 10\sum_{j=1}^{9} j + 45$$

$$= 46 + 8\binom{9+1}{3} + 10\frac{9 \times 10}{2}$$

$$= 1456$$

Hence the probability of a win is $1456/10000 = 0.1456 < 1/6$, and you shouldn't play this game!