COS 341, November 18, 1998
Handout Number 9

Solving Systems of Recurrences

Let a_n be the number of ways to tile with dominos a $3 \times n$ rectangle, and b_n be the number of ways to tile with dominos a $3 \times n$ quasi-rectangle (ie, rectangle with one corner square missing). From the discussions in class, we have $a_0 = 1, a_1 = 0, b_0 = 0, b_1 = 1$, and for all $n \geq 2$,

$$\begin{aligned} a_n &= a_{n-2} + 2b_{n-1}, \\ b_n &= a_{n-1} + b_{n-2}. \end{aligned} \tag{1}$$

Consider the generating functions $A(x) = \sum_{n \geq 0} a_n x^n$, and $B(x) = \sum_{n \geq 0} b_n x^n$. Then from (1) we obtain

$$\sum_{n \geq 2} a_n x^n = \sum_{n \geq 2} a_{n-2} x^n + 2 \sum_{n \geq 2} b_{n-1} x^n.$$

This leads to

$$A(x) - a_0 - a_1 x = x^2 A(x) + 2x(B(x) - b_0). \tag{2}$$

Similary, we obtain

$$B(x) - b_0 - b_1 x = x(A(x) - a_0) + x^2 B(x). \tag{3}$$

Substituting the values of a_0, a_1, b_0, b_1 into (2) and (3), we obtain after rearranging terms,

$$(1 - x^2)A(x) - 2xB(x) = 1, \tag{4}$$

$$xA(x) - (1 - x^2)B(x) = 0. \tag{5}$$

We now solve (4) and (5) for $A(x), B(x)$. From (5) we have

$$B(x) = \frac{x}{1 - x^2} A(x). \tag{6}$$

Substituting (6) into (4) we have

$$(1 - x^2)A(x) - 2x \frac{x}{1 - x^2} A(x) = 1.$$

This leads immediately to

$$A(x) = \frac{1 - x^2}{(1 - x^2)^2 - 2x^2}. \tag{7}$$

It remains to extract a_n from its generating function $A(x)$. Let $y = x^2$. We have

$$A(x) = \frac{1 - y}{1 - 4y + y^2}.$$

Using the same partial fraction decomposition method as before, we obtain

$$\begin{aligned} A(x) &= (1 - y) \frac{1}{(1 - (2 + \sqrt{3})y)(1 - (2 - \sqrt{3})y)} \\ &= (1 - y) \frac{1}{2\sqrt{3}} \left(\frac{2 + \sqrt{3}}{(1 - (2 + \sqrt{3})y} - \frac{2 - \sqrt{3}}{(1 - (2 - \sqrt{3})y} \right) \end{aligned}$$

$$
\begin{aligned}
&= (1-y)\frac{1}{2\sqrt{3}}\sum_{n\geq 0}((2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1})y^n \\
&= (1-x^2)\frac{1}{2\sqrt{3}}\sum_{n\geq 0}((2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1})x^{2n} \\
&= \frac{1}{2\sqrt{3}}\sum_{n\geq 0}((2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1})x^{2n} - \\
&\qquad \frac{1}{2\sqrt{3}}\sum_{n\geq 0}((2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1})x^{2(n+1)}.
\end{aligned}
$$

Since a_m is equal to the coefficient of the x^m term in the above expression, we conclude that for all $n \geq 0$,

$$
\begin{aligned}
a_{2n} &= \frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1}) - \frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n}-(2-\sqrt{3})^{n}), \\
a_{2n+1} &= 0.
\end{aligned}
$$

We have thus determined the number of ways to tile a $3 \times n$ rectangle using dominos.

You might be interested in calculating the value of a_4 using the above formula and compare it with the value obtained from the recurrence relations directly.