COS 341, Handout No. 7
November 16, 1998

Solutions to Midterm

Problem 1 The probability space is $U = \{H, T\}^n$. Consider the random variable X where, for each $u \in U$, $X(u)$ is equal to $7^{k(u)}$ where $k(u)$ is the number of HEADs in u. Then the answer we seek is

$$\begin{aligned} E(X) &= |U|^{-1} \sum_{u \in U} X(u) \\ &= 2^{-n} \sum_{0 \le j \le n} \sum_{u, k(u)=j} 7^j \\ &= 2^{-n} \sum_{0 \le j \le n} \binom{n}{j} 7^j \\ &= 2^{-n} f(7), \end{aligned}$$

where $f(x) = (1+x)^n = \sum_{0 \le j \le n} \binom{n}{j} x^j$. This immediately gives $E(X) = 4^n$.

Problem 2 This is equivalent to counting the number of ways the tycoon can give away n dollars to his six children without the \$10,000 requirement, where $n = 1,000,000 - 60,000 = 940,000$. Thus the number of ways is equal to

$$\binom{n+6-1}{6-1} = \binom{940,005}{5}.$$

Problem 3 (a) The answer A is equal to

$$\begin{aligned} \sum_{0 \le i \le n} \binom{n}{i} \frac{1}{i+1} &= \sum_{0 \le i \le n} \frac{n!}{i!(n-i)!} \frac{1}{i+1} \\ &= \sum_{0 \le i \le n} \frac{n!}{(i+1)!(n-i)!} \\ &= \frac{1}{n+1} \sum_{0 \le i \le n} \frac{(n+1)!}{(i+1)!((n+1)-(i+1))!} \\ &= \frac{1}{n+1} \left(\sum_{0 \le i \le n+1} \binom{n+1}{i} - 1 \right) \\ &= \frac{1}{n+1} (2^{n+1} - 1). \end{aligned}$$

(b)

$$\sum_{0 \le i \le n} \binom{n}{i} \frac{1}{i+2} = \sum_{0 \le i \le n} \binom{n}{i} \frac{(i+2)-1}{i+1} \frac{1}{i+2}$$

$$
\begin{aligned}
&= \sum_{0\le i\le n} \binom{n}{i} \left(\frac{1}{i+1} - \frac{1}{(i+1)(i+2)}\right) \\
&= A - \sum_{0\le i\le n} \binom{n}{i} \frac{1}{(i+1)(i+2)}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\sum_{0\le i\le n} \binom{n}{i} \frac{1}{(i+1)(i+2)} &= \sum_{0\le i\le n} \frac{n!}{(i+2)!(n-i)!} \\
&= \frac{1}{(n+1)(n+2)} \sum_{0\le i\le n} \frac{(n+2)!}{(i+2)!((n+2)-(i+2))!} \\
&= \frac{1}{(n+1)(n+2)} \left(\sum_{0\le i\le n+2} \binom{n+2}{i} - 1 - \binom{n+2}{1}\right) \\
&= \frac{1}{(n+1)(n+2)} (2^{n+2} - n - 3).
\end{aligned}
$$

This leads to the answer

$$
\frac{1}{n+1}(2^{n+1} - 1) - \frac{1}{(n+1)(n+2)}(2^{n+2} - n - 3).
$$

Problem 4 (a)We give a combinatorial proof of

$$
\sum_{0\le i\le n} \binom{n}{i} \binom{n}{n-i} = \binom{2n}{n}.
$$

The right-hand side is the number of size-n subsets of $\{1, 2, \cdots, 2n\}$, which can be counted by summing up N_i, where N_i is the number of such subsets with i elements chosen from $\{1, 2 \cdots, n\}$ and $n-i$ elements chosen from $\{n+1, \cdots, 2n\}$. But this is exactly the expression for the left-hand side.
(b)

$$
\begin{aligned}
\sum_{0\le i<j\le n} \binom{n}{i} \binom{n}{j} &= \frac{1}{2}\left(\sum_{0\le i\le n, 0\le j\le n} \binom{n}{i} \binom{n}{j} - \sum_{0\le i\le n} \binom{n}{i} \binom{n}{i}\right) \\
&= \frac{1}{2}\left(\left(\sum_{0\le i\le n} \binom{n}{i}\right)^2 - \sum_{0\le i\le n} \binom{n}{i} \binom{n}{n-i}\right) \\
&= \frac{1}{2}\left(2^{2n} - \binom{2n}{n}\right),
\end{aligned}
$$

where we have used (a) in the last step.