COS 341, October 28, 1998
Handout Number 5

Last Year's Midterm Solutions

Problem 1 There are four ways to cut off three corners of the board. Whether the resulting board can be covered perfectly does not depend on which way one chooses (any one can be rotated into another). Thus, it suffices to prove it for any particular choice of three corners.

Label the 81 cells as *cell*-(i, j) with $0 \le i, j \le 8$, with the SW-corner cell as the origin cell-$(0,0)$. We show that if cut off all the four corners except cell-$(8,8)$, the resulting board cannot be covered perfectly.

Color the cells with three colors $0,1,2$, with cell-(i,j) in color $(i+j)\mathrm{mod}3$. Then there are exactly 27 cells of each color. If we cut off the three corners cells-$(0,0),(0,8),(8,0)$, the resulting board will have 26 cells of color 0, 27 cells of color 1, and 25 cells of color 2. Since each trimino must occupy one cell of each color, this board cannot be covered perfectly.

Problem 2 From binomial theorem,

$$(1+x)^{2n} = \sum_{0\le m\le 2n} \binom{2n}{m} x^m.$$

Let $f(x) = ((1+x)^{2n} - (1-x)^{2n})/2$. Then

$$f(x) = \sum_{1\le k\le n} \binom{2n}{2k-1} x^{2k-1}. \tag{1}$$

(a) The sought-after answer is by (1) equal to

$$f(2) = (3^n - 1)/2.$$

(b) Let A be the answer. Let $g(x) = xf(x)$. Then from (1)

$$g'(x) = \sum_{1\le k\le n} \binom{2n}{2k-1} 2kx^{2k-1}.$$

and hence

$$xg'(x)/2 = \sum_{1\le k\le n} \binom{2n}{2k-1} k(x^2)^k.$$

Thus,

$$\begin{aligned} A &= \sum_{1\le k\le n} \binom{2n}{2k-1} k2^k \\ &= \sqrt{2}g'(\sqrt{2})/2. \end{aligned} \tag{2}$$

Now by definition

$$\begin{aligned} g'(x) &= f(x) + xf'(x) \\ &= \frac{(1+x)^{2n} - (1-x)^{2n}}{2} + x\frac{2n(1+x)^{2n-1} + 2n(1-x)^{2n-1}}{2}. \end{aligned}$$

This, together with (2), implies

$$A = \frac{1}{\sqrt{2}} \left(\frac{(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}}{2} + n\sqrt{2}\frac{(1+\sqrt{2})^{2n-1} + (1-\sqrt{2})^{2n-1}}{2} \right).$$

Problem 3

Solution 1 (a) From the recurrence, $a_1 = (1-1)a_0 + 1 = 1$. Let $b_n = na_n$ for $n \geq 1$. Then $b_1 = a_1 = 1$, and for $n > 1$ we have the recurrence $b_n = b_{n-1} + 1$. This leads to

$$b_n = b_{n-2} + 2 = ... = b_k + (n-k) = b_1 + (n-1) = n.$$

It follows that for $n \geq 1$ we have $a_n = b_n/n = 1$.

(b) From the recurrence, $a_1 = (1-2)a_0 + 1 = -2$, and $a_2 = ((2-2)a_1 + 1)/2 = 1/2$. The recurrence gives for $n > 0$

$$n(n-1)a_n = (n-1)(n-2)a_{n-1} + (n-1).$$

Let $b_n = n(n-1)a_n$ for $n \geq 1$. Then $b_1 = 0$, and for $n \geq 2$ we have the recurrence $b_n = b_{n-1} + (n-1)$. This leads to

$$\begin{aligned} b_n &= b_{n-1} + (n-1) \\ b_{n-2} + ((n-2) + (n-1)) & \\ &= ... \\ &= b_k + ((n-1) + (n-2) + ... + k) \\ &= b_1 + (n + (n-1) + (n-2) + ... + 2) \\ &= n(n-1)/2. \end{aligned}$$

It follows that for $n \geq 2$ we have $a_n = b_n/(n(n-1)) = 1/2$.

Solution 2 (a) Calculate the first few a_n, and see a pattern $a_0 = 3, a_1 = 1, a_2 = 1, a_3 = 1$. Conjecture that $a_n = 1$ for $n \geq 1$, and then prove by induction as follows. The base case $n = 1$ is clearly true since we have calculated $a_1 = 1$. For the inductive step let $m > 1$ and assume that we have proved $a_n = 1$ for all $1 \leq n < m$. Then the recurrence gives $a_m = ((m-1)a_{m-1} + 1)/m = ((m-1) + 1)/m = 1$. This completes the induction.

(b) Calculate the first few a_n, and see a pattern $a_0 = 3, a_1 = -2, a_2 = 1/2, a_3 = 1/2, a_4 = 1/2$. Conjecture that $a_n = 1/2$ for $n \geq 2$, and then prove by induction as follows. The base case $n = 2$ is clearly true since we have calculated $a_2 = 1/2$. For the inductive step let $m > 2$ and assume that we have proved $a_n = 1/2$ for all $2 \leq n < m$. Then the recurrence gives $a_m = ((m-2)a_{m-1} + 1)/m = ((m-2)/2 + 1)/m = 1$. This completes the induction.