COS 341, September 28, 1998

Handout Number 2

Proof of Ramsey's Theorem

Extended Ramsey's Theorem: Let $m, n \geq 2$ be integers. Then among any 2^{m+n} people, there exist either m mutual friends, or n mutual strangers.

Let $P(m, n)$ denotes the statement "Among any 2^{m+n} people, there exist either m mutual friends, or n mutual strangers".

To prove the above theorem by induction, let $Q(s)$ be the statement "$P(m, n)$ is true, for any integers $m, n \geq 2$ and $m + n = s$". It suffices to prove by induction on $s \geq 4$ the validity of the statement $Q(s)$.

The base case: $s = 4$, then $m = n = 2$. Among $2^{2+2} = 16$ people, take the first 2, then either they are friends or strangers. We have proved that $Q(4)$ is true.

Inductively, let $s > 4$. Assume that $Q(4), Q(5), \cdots, Q(s-1)$ are true. We show that $Q(s)$ is true. There are several cases.

case 1: $m = 2, n = s - 2$. Take the first n of the 2^{m+n} people (recall $2^n > n$). Either there is at least a pair of friends, or they are all mutual strangers, we have proved $P(m, n)$ in this case.

case 2: $n = 2, m = s - 2$. Similar to case 1.

case 3: $m, n > 2$. Assume that person 1 has x friends; the rest $y = 2^{m+n} - 1 - x$ are strangers to person 1. Clearly, either $x \geq 2^{m+n-1}$ or $y \geq 2^{m+n-1}$.

Assume the former case, with S being the set of friends to person 1. By induction hypothesis applied to $m - 1, n$, there are either $m - 1$ people in S that are mutual friends (in which case together with person 1 we get m mutual friends), *or* m people in S that are mutual strangers. Thus in both situations the conclusion needed for $P(m, n)$ is true.

The case $y \geq 2^{m+n-1}$ can be dealt with similarly. This completes the inductive step for proving $Q(s)$. We have completed the proof of the Extended Ramsey's Theorem.