COS 341, December 2, 1998
Handout Number 11

A Property of Planar Graphs

Fact 1 Let G be a connected planar graph with v vertices, e edges and f faces. Then $v - e + f = 2$.

Theorem Let G be a planar graph with $v \geq 3$ vertices and e edges. Then $e \leq 3v - 6$.

Proof As described in class, we can add edges to G to make it a triangulated (and still planar) graph. Call this new graph G. It has v vertices and $e' \geq e$ edges. We prove

$$e' = 3v - 6. \tag{1}$$

This implies the Theorem since $e \leq e'$.

Since G' is triangulated, each edge belongs to the boundary of exactly two faces, and each face has exactly three edges on its boundary. Let f' be the number of faces of G'. Then the number of all pairs of the form (x, y), where x is an edge on the boundary of face y, is equal to $2e'$ and also equal to $3f'$. Thus,

$$2e' = 3f'. \tag{2}$$

Now note that by Fact 1 we have

$$v - e' + f' = 2. \tag{3}$$

Multiply both sides of (3) by a factor 3, and add to (2), we can cancel out f' to obtain

$$3v - e' = 6.$$

This proves (1) and hence the Theorem.