COS 341, Autumn 1998

Solutions to the Final Exam

Problem 1

(a) TRUE. By Ramsey's Theorem, among 4^k people there must be either k mutual friends or k mutual strangers. Take $k = \lfloor \log_4 n \rfloor \geq \log_4 n - 1$. Then, any graph on n $(\geq 4^k)$ vertices must either have a clique of size k or an independent set of size k.

(b) TRUE. Since $\deg(x)+\deg(y) \geq n$ for all vertices x, y, the graph G has the Ore's property and hence has a Hamiltonian cycle. Thus, any two distinct vertices u, v have two paths connecting them whose lengths add up to exactly n. One of these two paths must have length at most $n/2$.

(c) TRUE. Clearly, the statement is true for $n = 1, 2$. Let G_n denote the graph described with n vertices. We prove by induction on $n \geq 3$ that there is a planar embedding of G_n such that one of its faces (call it F_n) has as its boundary the three vertices $n, n-1, n-2$ and the three edges among them. The base case $n = 3$ is clearly true. In the inductive step, just put the new vertex $n+1$ inside the face F_n, and connect vertices $n, n-1, n-2$ to the new vertex. Note that we now also have a new face F_{n+1} bounded by $n+1, n, n-1$ (and the edges among them). This completes the inductive step.

Problem 2

We assume that n is divisible by 6, as stated in the problem statement. When n is not divisible by 6, some modifications in the derivation and the final answer are necessary.

Let W be the set of all triplets of integers (i, j, k) satisfying $i, j, k \geq 0$ and $i + j + k \leq n$. Let S_1 be the set of all $(i, j, k) \in W$ such that $i = j$, let S_2 be the set of all $(i, j, k) \in W$ such that $j = k$, and let S_3 be the set of all $(i, j, k) \in W$ such that $i = k$.

Clearly,

$$a_n = |W| - |S_1 \cup S_2 \cup S_3|. \tag{1}$$

Since $|W|$ is equal to the number of quadruples of integers (i, j, k, h) satisfying $i, j, k, h \geq 0$ and $i + j + k + h = n$, we have

$$|W| = \binom{n+3}{3}. \tag{2}$$

We now evaluate $|S_1 \cup S_2 \cup S_3|$. By Inclusion-Exclusion,

$$\begin{aligned} |S_1 \cup S_2 \cup S_3| &= |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| \\ &\quad - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|. \end{aligned} \tag{3}$$

Clearly,

$$\begin{aligned} |S_3| = |S_2| = |S_1| &= |\{(i,k) | i, k \geq 0, 2i + k \leq n\}| \\ &= \sum_{0 \leq i \leq n/2} (n - 2i + 1) \end{aligned}$$

$$
\begin{aligned}
&= (n+1)(1+n/2) - 2\sum_{0\le i\le n/2} i \\
&= (n+1)(1+n/2) - 2(1/2)(n/2)(1+n/2) \\
&= (1+n/2)^2. \qquad (4)
\end{aligned}
$$

Also,

$$
\begin{aligned}
|S_1\cap S_2\cap S_3| = |S_2\cap S_3| = |S_1\cap S_3| = |S_1\cap S_2| &= |\{(i,j,k)|i,j,k\ge 0, i=j=k, i+j+k\le n\}| \\
&= |\{i|i\ge 0, 3i\le n\}| \\
&= 1+n/3. \qquad (5)
\end{aligned}
$$

It follows from (3)-(5) that

$$
\begin{aligned}
|S_1\cup S_2\cup S_3| &= 3(1+\frac{n}{2})^2 - 3(1+\frac{n}{3}) + (1+\frac{n}{3}) \\
&= 3(1+\frac{n}{2})^2 - 2(1+\frac{n}{3}). \qquad (6)
\end{aligned}
$$

¿From (1),(2) and (6), we obtain

$$
a_n = \binom{n+3}{3} - 3(1+\frac{n}{2})^2 + 2(1+\frac{n}{3}).
$$

Problem 3

(a) From the "Remarks" in the exam problem sheet, $b_4 = 4$. We now determine b_5 by enumerating strings σ of length 5 that contain 101 a substring. (i) For σ with prefix 0, there are b_4 such strings; (ii) For σ with prefix 100, there is no such string; (iii) For σ with prefix 101, there are 4 such strings (ie, $101**$); (iv) For σ with prefix 110, there are 2 such strings (ie, $1101*$); (iii) For σ with prefix 111, there is 1 such string (ie, 11101). All together, $b_5 = b_4 + 0 + 4 + 2 + 1 = 11$.

(b) Let d_n (e_n) be the number of binary strings of length n that begin with the bit 0 (1) and that contain 101 as a substring. Then,

$$d_1 = e_1 = d_2 = e_2 = 0.$$

For $n \ge 3$, note that the number of binary strings of length n with prefix 00 and containing 101 as a substring is equal to d_{n-1}, and the number of such strings with prefix 01 is equal to e_{n-1}; Thus, $d_n = d_{n-1} + e_{n-1}$. Similarly, the number of such strings with prefix 11 is equal to e_{n-1}, that with prefix 100 is equal to d_{n-2}, and that with prefix 101 is equal to 2^{n-3}. This gives $e_n = e_{n-1} + d_{n-2} + 2^{n-3}$. In summary, for $n \ge 3$,

$$
\begin{aligned}
d_n &= d_{n-1} + e_{n-1}, \\
e_n &= e_{n-1} + d_{n-2} + 2^{n-3}. \qquad (7)
\end{aligned}
$$

Also note that, for $n \ge 1$,

$$b_n = d_{n+1}. \qquad (8)$$

Let $D(x) = \sum_{n\geq 1} d_n x^n$, and $E(x) = \sum_{n\geq 1} e_n x^n$. By definition, $B(x) = \sum_{n\geq 1} b_n x^n$. Using (8) and the fact $d_1 = 0$ we have

$$B(x) = (1/x) \sum_{n\geq 1} d_{n+1} x^{n+1} = (1/x) D(x). \tag{9}$$

Thus, to obtain an expression for $B(x)$, it suffices to evaluate $D(x)$.

¿From (7)

$$\sum_{n\geq 3} d_n x^n = \sum_{n\geq 3} d_{n-1} x^n + \sum_{n\geq 3} e_{n-1} x^n.$$

This gives

$$D(x) - d_1 x - d_2 x^2 = x(D(x) - d_1 x) + x(E(x) - e_1 x).$$

Substituting the values of d_1, d_2, e_1 and rearranging terms, we obtain

$$(1-x)D(x) - xE(x) = 0. \tag{10}$$

Similarly, we can obtain from the other formula in (7) that

$$x^2 D(x) - (1-x)E(x) = -\frac{x^3}{1-2x}. \tag{11}$$

Now we can eliminate $E(x)$ from (10) and (11) to obtain

$$D(x) \quad = \quad \frac{x^4}{(1-2x)((1-x)^2 - x^3)}. \tag{12}$$

Substituting (12) into (9), we otain

$$B(x) = \frac{x^3}{(1-2x)((1-x)^2 - x^3)}.$$