COS 341, Autumn 1998
January 5, 1999

Final Exam

Important Note that the due date has been extended to 11am on January 8. Be sure to write down your name on the first page of your submitted solutions.

This is a take-home open-book exam. Do all three problems. No collaboration is allowed. You should only work during a 24-hour (consecutive) period. Please hand in your solutions to Ms. Sandy Barbu in Room 323 in the Computer Science Department, by 11:00 am on January 8, 1999. If you have any question, send me e-mail at yao@cs.princeton.edu.

Problem 1 [30 points] For each of the following statements, give a "True" or "False" answer. Justify your answers concisely.
(a) [10 points] Let n be any positive integer. Any graph G on n vertices must satisfy either $\omega(G) \geq (\log_4 n) - 1$ or $\alpha(G) \geq (\log_4 n) - 1$. (Recall that $\omega(G)$ is the maximum size of any clique of G, and that $\alpha(G)$ is the maximum size of any independent set of G.)
(b) [10 points] Let n be any positive integer. Let G be a graph on n vertices satisfying $\deg(v) \geq n/2$ for all vertices v. Then between any two distinct vertices u and v, there exists in G a path of length at most $n/2$ that connects u and v.
(c) [10 points] Let n be any positive integer. Let $G = (V, E)$, where $V = \{1, 2, \cdots, n\}$ and E is the set of all $\{i, j\}$ $(1 \leq i < j \leq n)$ satisfying $0 < |i - j| \leq 3$. Then G is a planar graph.

Problem 2 [25 points] Let $n = 6s$ where s is a positive integer. Let a_n be the number of triplets of integers (i, j, k) such that (1) i, j, k are non-negative and distinct integers (ie, $i \geq 0, j \geq 0, k \geq 0, i \neq j, j \neq k, i \neq k$), and (2) $i + j + k \leq n$. Derive an explicit closed-form expression for a_n.

Problem 3 [25 points] Let $n \geq m > 0$ be integers. A binary string $u = u_1 u_2 \cdots u_n$ is said to *contain* $v = v_1 v_2 \cdots v_m$ *as a substring* if there exists an integer i (where $0 < i \leq n - m + 1$) such that $u_i = v_1, u_{i+1} = v_2, \cdots, u_{i+m-1} = v_m$. For example, 110100 contains 101 as a substring (with $i = 2$), while the string 001100 does not contain 101 as a substring.

Let b_n denote the number of binary strings of length n that contain 101 as a substring. Clearly, $b_1 = b_2 = 0$. Let $B(x) = \sum_{n \geq 1} b_n x^n$.
(1) [5 points] Determine the value of b_5.
(2) [20 points] Derive an explicit closed-form expression for $B(x)$.

Hint You might want to first set up recurrence relations for the appropriate sequences.

Remarks For $n = 3$, there is only one binary string of length n that contains 101 as a substring (ie, 101). For $n = 4$, there are exactly four binary strings of length n that contain 101 as a substring (ie, $0101, 1010, 1011, 1101$). Thus, $b_3 = 1, b_4 = 4$.