Solutions to Homework Set 7

Section 12.6, Problem 5

Call the three given graphs G_1, G_2 and G_3 from left to right. We see that G_1 contrains K_2 as a subgraph, G_2 contains K_3 and G_3 contains K_4. This means $\chi(G_1) \geq 2, \chi(G_2) \geq 3, \chi(G_3) \geq 4$. By a little trial-and-error it is easy to come up with a 2-coloring of G_1, a 3-coloring of G_2 and a 4-coloring of G_3. Thus, $\chi(G_1) = 2, \chi(G_2) = 3$ and $\chi(G_3) = 4$. □

Section 12.6, Problem 13

This is a simple application of induction. Let q_n be chromatic polynomial for an n-cycle. If $n = 3$, we can explicitly count the number of k-colorings of a 3-cycle to obtain

$$q_3(k) = k(k-1)(k-2) = (k-1)^3 + (-1)^3(k-1).$$

Now suppose $n \geq 3$ and consider an $(n+1)$-cycle G. By contracting an edge of G we obtain an n-cycle G_1 and by deleting an edge of G we obtain a path G_2 of length $n+1$, which is a tree with $n+1$ nodes. Therefore $p_{G_2}(k) = k(k-1)^n$. By a theorem in the textbook, we have

$$q_{n+1}(k) = p_G(k) = p_{G_2}(k) - p_{G_1}(k) = k(k-1)^n - q_n(k).$$

Applying the induction hypothesis gives

$$q_{n+1}(k) = k(k-1)^n - ((k-1)^n + (-1)^n(k-1)) = (k-1)^{n+1} + (-1)^{n+1}(k-1)$$

and the result is proved. □

Section 11.9, Problem 18

The simplest example of a planar graph with chromatic number 4 that doesn't contain a K_4 is the *5-wheel*. For integers $n \geq 3$, an *n-wheel* is defined as an n-cycle with one special vertex added; the special vertex is adjacent to all n vertices forming the cycle.

An n-wheel is clearly planar, 4-colorable and, if $n \geq 4$, does not contain a K_4. Why is a 5-wheel not 3-colorable? Since the "rim" of the wheel (the 5-cycle) is an *odd cycle* it requires at least three colors. The color of the special (the 6th) vertex must be different from that of any of the vertices on the cycle. Hence the result. □

Section 11.9, Problem 19

Proof by induction: If we have just one line in the plane then we can clearly do the required 2-coloring by just coloring the two regions thus created differently. Now suppose the set of regions of a plane created by n dividing lines is 2-colorable. Consider adding an $(n+1)^{st}$ line to a properly colored plane divided by n lines. The new line splits the entire plane into two half-planes, each of which is correctly colored. Flip the color of every region in one of the half-planes. This still keeps each half-plane correctly colored. But now if two regions share the $(n+1)^{st}$ line for a boundary, they are differently colored, because

- before adding the $(n+1)^{st}$ line they must have been one region
- thus, before the color flipping they must have been colored the same.

Thus, after the flipping, the entire plane is correctly colored. □

Direct proof: This proof relies less on geometric intuition but one needs to work a little to make it rigorous. Consider a graph whose vertex set is the set of regions formed by the n lines and where two vertices are adjacent iff the corresponding regions share a boundary in the plane.

We must show that this graph is 2-colorable, i.e., bipartite, i.e., that every cycle in it is of even length.

Each of the n-lines divides the plane into two half-planes; arbitrarily associate the bit "0" with one of them (half-planes) and the bit "1" with the other. Now every region in the plane (every vertex in the graph) can be associated with an n-bit string where the i^{th} bit says whether the region lies in the "0-half-plane" or the "1-half-plane" associated with the i^{th} line. Identify each vertex with its associated bit string.

Two adjacent bit strings (vertices) clearly differ in exactly one bit position. But if we have a cycle of length k in the graph, then traversing around the cycle we should see exactly k bit-flips. However, we must end up with the same bit string we started with; hence each bit must get flipped an even number (possibly 0) of times. Hence k must be even. $\square$

Section 11.9, Problem 30

A tree contains no cycles; in particular, no cycles of odd length. Therfore *a tree is bipartite.* Let T be a tree, $\chi(T) \leq 2$. But $\chi(T) \geq n/\alpha(T)$. Therefore $\alpha(T) \geq \lceil n/\chi(T) \rceil \geq \lceil n/2 \rceil$. $\square$

Note that a null graph is also a tree, so the chromatic number of a tree could be 1. It is a subtle point, but it cannot be neglected.

Section 11.9, Problem 42

An induced sugraph H of a bipartite graph G is bipartite. Therefore, H has no odd cycles; in particular, no triangles. Therefore $\omega(H) \leq 2$. By bipartiteness, $\chi(H) \leq 2$. Now suppose $\omega(H) = 1$. Then H must be a null graph and so $\chi(H) = 1$. If $\omega(H) = 2$ then, since $\chi(H) \geq \omega(H)$ and $\chi(H) \leq 2$, we get $\chi(H) = 2$. Thus, in any case, $\omega(H) = \chi(H)$.

Since H was an *arbitrary* induced subgraph, this is true of *all* induced subgraphs of G. Therefore, G is χ-perfect; hence perfect. $\square$

Special Problem 1

This was a completely mechanical problem, so I've given credit for wrong final answers which were wrong only because of some "careless copying errors". The graph G_S which you were asked to construct has 12 vertices. After some trial and error you'll see that it has exactly *three* Eulerian trails, all starting at vertex GAC and ending at CGT. Following the Eulerian trails gives you three possible $\overline{\sigma}$'s, hence three possible σ's:

$$CTGCTTGACTGAGTGCA$$

$$CTGACTCGTTGAGTGCA$$

$$CTGAGTGCTTGACTGCA$$

and it's now easy to infer that the dinosaur is vegetarian and can fly but may not be timid.

The edges of G_S can be partitioned into either three or two edge-disjoint cycles. The former partitioning gives a G'_S which is a triangle while the latter gives a G'_S which has two vertices connected by three parallel edges. $\square$

Special Problem 2

Given a graph G on n vertices, we perform the following reduction: add a special vertex u_1 and connect it to every vertex in G. Form an n-wheel with $(n+1)$ new vertices whose special (central) vertex we call u_2. Now connect up u_1 and u_2 using more new vertices and edges according to the way we defined "handlebars" for the bicycle. (This construction can be made rigorous, but it's hardly the important part!) Let H be the graph so formed.

Our input to the $BICYCLE$ problem will be the graph H and the number n. (**Important: *almost all of you forgot that the input to $BICYCLE$ is not just a graph!***)

The number of edges and vertices of H are clearly polynomials in n; therefore this is a polynomial time reduction. (Actually we need to argue that we can *construct* H in polynomial time, but let that be assumed.)

If G has a Hamilton cycle then the grpah H contains a subgraph which is a bicycle of order n just because of the way we constructed it. Simply remove from H all those edges of G which do not appear in some (fixed) Hamilton cycle of G and you're left with a bicycle of order n.

If H has a subgraph which is a bicycle of order n, then how do we know that G has a Hamilton cycle? We know because this bicycle in H must contain two "hubs" (the vertices from which the "spokes" of the "wheels" emanate) and *since the bicycle is of order* n these hubs must have degree $(n+1)$ each. Now, by construction, only two vertices in H have that degree: namely, u_1 and u_2. Consider the wheel with hub u_1; it is a cycle of length n and lies completely in G. But n is the number of vertices of G! Therefore, this cycle is a Hamilton cycle of G. $\square$

Comments? Corrections? Suggestions? Mail to `amitc@cs.princeton.edu`