Solutions to Homework Set 6

Section 11.9, Problem 9

How do I draw a graph here?!!! Anyway, this problem was very easy and almost evryone got it right. Those who didn't forgot that in a *graph* (as opposed to a *general graph*) multiple edges are not allowed between any pair of vertices.

Section 11.9, Problem 10

Very few people gave a fully rigorous argument for this.

Suppose G is a connected graph with $n \geq 3$ vertices with degree sequence $(2, 2, \ldots, 2)$. Choose any vertex of G and call it x_1. Since $\deg x_1 = 2$, there is a vertex x_2 adjacent to x_1. Since $\deg x_2 = 2$, there is a vertex $x_3 \neq x_1$ adjacent to x_2. Continue this process to get a sequence $x_1, x_2, x_3, \ldots$ of vertices of G.

But G has only n vertices; so the elements of this sequence cannot be distinct. Suppose x_{k+1} is the first repetition in the sequence and $x_{k+1} = x_j$ for some $j < k - 1$. If $j > 1$, $\deg x_j \geq 3$ and we have a contradiction. If $k < n$, we already have two edges incident to each of $x_1, \ldots, x_k$ and so these vertices are disconnected from the rest of the graph, again a contradiction.

Thus $k = n$, $j = 1$ and the graph G is an n-cycle. That all n-cycles are isomorphic is the easy part; just label vertices in cyclic order. $\square$

There's another really neat way of proving this: Since G is connected and every vertex has even degree it has an Eulerian trail T. In T, each vertex v of G appears exactly $\deg v/2$ times. But from the given degree sequence it follows that each vertex appears exactly once! Therefore T is a cycle and since T contains every edge of G, G is a cycle too. $\square$

Section 11.9, Problem 12

The leftmost graph is isomorphic to the middle graph and the only way to show this is by explicitly giving labelings for the two graphs such that (i, j) is an edge in the first iff (i, j) is an edge in the second. Fortunately, many such labelings exist and it is easy to find one.

These two graphs have no 4-cycles. However the rightmost graph has one (well, at least one)! Therefore it cannot be isomorphic to either of the first two graphs. $\square$

Section 11.9, Problem 20

Once again, lots of bogus arguments were submitted for this problem.

Surprisingly enough, only one student found the *induction proof* which I think is the simplest. The statement is trivial for $n = 2$. Let $n \geq 2$. Let G be graph of order $(n + 1)$ with at least $n(n - 1)/2 + 1$ vertices. Suppose G is not connected. G cannot contain an isolated vertex, for if it did, it could have had at most $\binom{n}{2} = n(n - 1)/2$ edges (at most one edge between each pair amongst the other n vertices).

So we can delete a vertex x from G and obtain a non-connected graph $G - x$ with n vertices. Since G was not connected, clearly no vertex had degree n in G. Thus $\deg x \leq n - 1$, the number of edges in $G - x$ is at least $n(n - 1)/2 + 1 - (n - 1) = (n - 1)(n - 2)/2 + 1$. But then, by the induction hypothesis $G - x$ must be connected, a contradiction. $\square$

As for the example of an n-vertex graph with $(n - 1)(n - 2)/2$ edges that is not connected, simply consider K_{n-1} with one isolated vertex added. **Important:** *it is not enough to give an example for, say, $n = 5$.*

Section 11.9, Problem 40

We will prove that G satisfies the Ore property.

If G is complete, then it clearly has a Hamilton cycle, so we need not consider this case anymore. So let u, v be two non-adjacent vertices of G. There can be at most $\binom{n-2}{2} = (n-2)(n-3)/2$ edges amongst the vertices of G other than u, v. Each of the remaining edges is incident to either u or v, so it contributes 1 to the sum $\deg u + \deg v$. Thus

$$\deg u + \deg v \geq \frac{(n-1)(n-2)}{2} + 2 - \frac{(n-2)(n-3)}{2} = n.$$

Since u and v were *any two* non-adjacent vertices, we have verified the Ore property. $\square$

As for the example of an n-vertex graph with $(n-1)(n-2)/2 + 1$ edges that does not have a Hamilton cycle, simply consider K_{n-1} with one extra vertex joined to it by a single edge. Since this "single edge" is a bridge in the graph, the graph cannot have a Hamilton cycle.

Problems on Prüfer's algorithm

These were completely mechanical and were solved correctly by almsot everyone.

Comments? Corrections? Suggestions? Mail to `amitc@cs.princeton.edu`