Solutions to Homework Set 5

Section 7.8, Problem 1

Let

$$\begin{aligned} a_n &= f_1 + f_3 + \cdots + f_{2n-1} \\ b_n &= f_0 + f_2 + \cdots + f_{2n} \\ c_n &= f_0 - f_1 + f_2 - f_3 + \cdots + (-1)^n f_n \\ d_n &= f_0^2 + f_1^2 + \cdots + f_n^2 \end{aligned}$$

By evaluating these for a few small values of n we can conjecture that

$$\begin{aligned} a_n &= f_{2n} \\ b_n &= f_{2n+1} - 1 \\ c_n &= (-1)^n f_{n-1} - 1 \quad \text{for } n > 0 \\ d_n &= f_n f_{n+1} \end{aligned}$$

These conjectures are easily proven by induction. Verifying the base cases is straightforward. Assuming the conjectures true for $n = k$ we show that they hold for $n = k + 1$ as follows:

$$a_{k+1} = a_k + f_{2k+1} = f_{2k} + f_{2k+1} = f_{2k+2}$$

$$b_{k+1} = b_k + f_{2k+2} = f_{2k+1} - 1 + f_{2k+2} = f_{2k+3} - 1$$

$$c_{k+1} = c_k + (-1)^{k+1} f_k = (-1)^k (f_{k-1} - f_{k+1}) - 1 = (-1)^{k+1} f_k - 1$$

$$d_{k+1} = d_k + f_{k+1}^2 = f_k f_{k+1} + f_{k+1}^2 = (f_k + f_{k+1}) f_{k+1} = f_{k+1} f_{k+2}$$

□

Section 7.8, Problem 7

Suppose $n > 2$. If we color the first cell white or blue then we may color the rest of the board in h_{n-1} ways in each case. If, instead, we color it red then we are forced to color the second cell either white or blue. In each of these subcases we may color the remaining $n - 2$ cells in h_{n-2} ways. Thus

$$h_n = 2h_{n-1} + 2h_{n-2}.$$

We also note that $h_1 = 3$ and $h_2 = 8$ by simply counting all colorings.

This is a linear homogeneous recurrence and thus we solve it using the standard procedure described in the text. The characteristic equation $x^2 - 2x - 2 = 0$ has roots $1 \pm \sqrt{3}$. Thus the general solution is

$$h_n = c_1(1 + \sqrt{3})^n + c_2(1 - \sqrt{3})^n.$$

Using the initial values $h_1 = 3$ and $h_2 = 8$ we can solve for c_1 and c_2. Doing so gives

$$c_1 = \frac{\sqrt{3} + 2}{2\sqrt{3}}, \quad c_2 = \frac{\sqrt{3} - 2}{2\sqrt{3}}.$$

□

Section 7.8, Problem 11

The recurrence $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$ has characteristic equation $x^3 - x^2 - 9x + 9 = 0$ whose roots are 1, 3 and -3. Thus the general solution is

$$h_n = c_1(1)^n + c_2(3)^n + c_3(-3)^n = c_1 + c_2 3^n + c_3(-3)^n.$$

Using the initial values $h_0 = 0$, $h_1 = 1$ and $h_2 = 2$ gives us three simultaneous linear equations for c_1, c_2, c_3 which we solve to get

$$c_1 = -1/4, \quad c_2 = 1/3, \quad c_3 = -1/12$$

and plugging these in the general solution gives

$$h_n = -\frac{1}{4} + 3^{n-1} + \frac{1}{4}(-3)^{n-1}.$$

□

Special Problem 1

Using the identities

$$\begin{aligned}
\sum_{j=0}^{N} \binom{N}{j} &= 2^N \\
\sum_{j=0}^{N} j\binom{N}{j} &= N \cdot 2^{N-1} \\
\sum_{j=0}^{N} j^2\binom{N}{j} &= N(N+1) \cdot 2^{N-2}
\end{aligned}$$

we easily get

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{m=0}^{n}(k+m)^2\binom{n}{k}\binom{n}{m} &= \sum_{k=0}^{n}\sum_{m=0}^{n}\left(k^2\binom{n}{k}\binom{n}{m} + 2km\binom{n}{k}\binom{n}{m} + m^2\binom{n}{k}\binom{n}{m}\right) \\
&= \sum_{k=0}^{n} k^2\binom{n}{k}\sum_{m=0}^{n}\binom{n}{m} + 2\sum_{k=0}^{n} k\binom{n}{k}\sum_{m=0}^{n} m\binom{n}{m} + \\
&\quad + \sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{n} m^2\binom{n}{m} \\
&= n(n+1)2^{n-2}2^n + 2(n2^{n-1})^2 + 2^n n(n+1)2^{n-2} \\
&= n(2n+1)2^{2n-1}.
\end{aligned}$$

□

Special Problem 2

There are many ways to solve this problem and many correct solutions were received too. At the outset we observe that a we may never move west if we are following a legal path.

METHOD ONE: At each N-S street except the last you have to make a decision: to continue on the E-W street you are currently on or to switch to the other E-W street. To choose a path from Q to W we thus have to make $n-1$ decisions with 2 choices each time. Thus, we may choose a path in 2^{n-1} ways and so there are 2^{n-1} paths from Q to W.

METHOD TWO: A path from Q to W is uniquely defined by the set of N-S streets it traverses. This set must be of odd size since we eventually must end up at a different E-W street from the one we started on. Thus, the number of paths equals the number of odd-sized subsets of an n-element set which, of course, equals 2^{n-1}.

METHOD THREE: Let R be the point directly north of Q. Let $h(n)$ denote the number of ways to get from R to W without traversing a segment twice. In a path from Q to W we may start

either by moving north, then east, and completing the path in one of $h(n-1)$ ways or by moving east and completing the path in one of $g(n-1)$ ways. Thus

$$g(n) = g(n-1) + h(n-1), \quad \text{for } n > 2.$$

A similar argument shows that

$$h(n) = g(n-1) + h(n-1), \quad \text{for } n > 2.$$

Thus $g(n) = h(n)$ for $n > 2$ and we easily see that this holds for $n = 2$ as well. Therefore, $g(n) = 2g(n-1)$ and $g(2) = 2$, whence $g(n) = 2^{n-1}$.

□

Special Problem 3

The given recurrence is not linear homogeneous and so we cannot apply our standard method directly. However, if we define $c_n = 1/d_n$ we have

$$c_n = \frac{6d_{n-1} + d_{n-2}}{d_{n-1}d_{n-2}} = c_{n-1} + 6c_{n-2}.$$

Now this *is* a a linear homogeneous recurrence, so we *can* apply our standard method. The characteristic equation $x^2 - x - 6 = 0$ has roots -2 and 3, whence the general solution is: $c_n = a_1(-2)^n + a_2 3^n$. From the initial values $d_0 = d_1 = 1$ we get $c_0 = c_1 = 1$. Plugging these in the general solution and solving for a_1, a_2 gives

$$a_1 = 2/5. \quad a_2 = 3/5.$$

And now,

$$c_n = \frac{2}{5}(-2)^n + \frac{3}{5}3^n = \frac{3^{n+1} - (-2)^{n+1}}{5}$$

$$d_n = \frac{1}{c_n} = \frac{5}{3^{n+1} - (-2)^{n+1}}.$$

□