Solutions to Homework Set 4

Problems from Section 5.7

Problem 34.

From the multinomial theorem we know that the term involving $x_1^3 x_2^3 x_3 x_4^2$ is:

$$\binom{9}{3 \quad 3 \quad 1 \quad 2}(x_1)^3(-x_2)^3(2x_3)(-2x_4)^2$$

So the coefficient of $x_1^3 x_2^3 x_3 x_4^2$ is:

$$\frac{9!}{3!3!1!2!} * 1 * (-1) * 2 * 4 = -40320$$

Problem 35.

Using the binomial theorem (twice):

$$(x_1 + x_2 + x_3)^n = ((x_1 + x_2) + x_3)^n = \sum_{k=0}^{n} \binom{n}{k} (x_1 + x_2)^k x_3^{n-k}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} x_1^j x_2^{k-j} x_3^{n-k}$$

Put $i_1 = j$, $i_2 = k - j$ and $i_3 = n - k$. Note that $0 \leq i_1, i_2, i_3 \leq n$ and $i_1 + i_2 + i_3 = n$. Thus the last expression simplifies to:

$$\sum_{i_1+i_2+i+3=n} \binom{n}{i_1 \quad i_2 \quad i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}$$

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Problems from Section 6.6

Problem 2.

First we note that the total number of integers between 1 and $10,000$ is 10000. From this we must subtract the total number of integers which are divisible by 4, 6, 7, or 10. Note the following:

$\lfloor 10000/4 \rfloor = 2500$ integers divisible by 4
$\lfloor 10000/6 \rfloor = 1666$ integers divisible by 6
$\lfloor 10000/7 \rfloor = 1428$ integers divisible by 7
$\lfloor 10000/10 \rfloor = 1000$ integers divisible by 10
$\lfloor 10000/12 \rfloor = 833$ integers divisible by 4 and 6, since $lcm\{4,6\} = 12$
$\lfloor 10000/28 \rfloor = 357$ integers divisible by $lcm\{4,7\} = 28$
$\lfloor 10000/20 \rfloor = 500$ integers divisible by $lcm\{4,10\} = 20$
$\lfloor 10000/42 \rfloor = 238$ integers divisible by $lcm\{6,7\} = 42$
$\lfloor 10000/30 \rfloor = 333$ integers divisible by $lcm\{6,10\} = 30$
$\lfloor 10000/70 \rfloor = 142$ integers divisible by $lcm\{7,10\} = 70$
$\lfloor 10000/84 \rfloor = 119$ integers divisible by $lcm\{4,6,7\} = 84$
$\lfloor 10000/60 \rfloor = 166$ integers divisible by $lcm\{4,6,10\} = 60$
$\lfloor 10000/140 \rfloor = 71$ integers divisible by $lcm\{4,7,10\} = 140$
$\lfloor 10000/210 \rfloor = 47$ integers divisible by $lcm\{6,7,10\} = 210$
$\lfloor 10000/420 \rfloor = 23$ integers divisible by $lcm\{4,6,7,10\} = 420$

The total number of integers not divisible by these numbers is, by the inclusion-exclusion principle, the total number of integers minus the number divisible by any one plus the number divisible by any two minus the number divisible by any three plus the number divisible by all four. Thus, the number of integers between 1 and $10,000$ not divisible by 4, 6, 7, or 10 is:

$$10000 - 2500 - 1666 - 1428 - 1000 + 833 + 357 + 500 + 238 + 333 + 142 - 119 - 166 - 71 - 47 + 23$$

$$= 5429$$

9.

The number of solutions is clearly equal to the number of solutions which satisfy the lower bounds minus those which do not satisfy the upper bounds. To find this, we apply the inclusion-exclusion principle.

First we find the number of solutions to the problem $x_1 + x_2 + x_3 + x_4 = 20, \quad 1 \leq x_1, 0 \leq x_2, 4 \leq x_3, 2 \leq x_4$ which is $\binom{13+3}{3} = \binom{16}{3} = 560$. Look at Homework Set 2 to find out how to obatin this.

Next, we find the number of solutions to each of the problems where one of the upper bounds is violated (and the others are satisfied):

$$\begin{array}{ll} 7 \leq x_1 & \Rightarrow \binom{7+3}{3} = \binom{10}{3} = 120 solutions \\ 8 \leq x_2 & \Rightarrow \binom{5+3}{3} = \binom{8}{3} = 56 solutions \\ 9 \leq x_3 & \Rightarrow \binom{8+3}{3} = \binom{11}{3} = 165 solutions \\ 7 \leq x_4 & \Rightarrow \binom{8+3}{3} = \binom{11}{3} = 165 solutions \end{array}$$

Next we find those situations where two of the upper bounds are violated and the remaining two equations satisfy their lower bounds:

$$\begin{array}{lll} 7 \leq x_1, & 8 \leq x_2 & \Rightarrow 0 solutions \\ 7 \leq x_1, & 9 \leq x_3 & \Rightarrow \binom{5}{3} = 10 solutions \\ 7 \leq x_1, & 7 \leq x_4 & \Rightarrow \binom{5}{3} = 10 solutions \\ 8 \leq x_2, & 9 \leq x_3 & \Rightarrow \binom{3}{3} = 1 solutions \\ 8 \leq x_2, & 7 \leq x_4 & \Rightarrow \binom{3}{3} = 1 solutions \\ 9 \leq x_3, & 7 \leq x_4 & \Rightarrow \binom{6}{3} = 20 solutions \end{array}$$

It is easy to see that there are no situations where three of the upper bounds are violated. Applying inclusion-exclusion, our answer is:

$$560 - 120 - 56 - 165 - 165 + 0 + 10 + 10 + 1 + 1 + 20 = 96$$

Problem 14.

There are $\binom{n}{k}$ ways to choose the k integers retaining their natural positions. The remaining $n-k$ integers must be "deranged" since none of them is to retain its position. There are D_{n-k} ways to do this. Thus, the total number of ways to arrange n integers is:

$$\binom{n}{k} D_{n-k}$$

where D_m is the number of ways one can form a derangement of m integers.

Problem 29.

This is yet another application of the inclusion-exclusion principle. Assume for now that we have a linear permutation of 8 boys. The total number of seating arrangements is thus $8! = 40320$.

We subtract off the total number of seating arrangements in which (at least) one pair of originally facing boys is still sitting across from each other. There are $\binom{4}{1}$ ways to pick the pair of boys, and 4 ways to pick the pair of seats for the boys to sit in, and 2^1 additional seating arrangements (from exchanging the position of the boys). Thus, there are $\binom{4}{1} * 4 * 2^1 = 32$ ways to seat a single pair of boys. The other 6 boys may sit wherever they wish, giving $32 * 6!$ as the total number of ways to seat boys with at least one pair retaining partners.

Now we follow a similar procedure for adding back the number of seating arrangements in which (at least) two pairs of originally facing boys are still sitting across from one another. There are $\binom{4}{2}$ ways to pick the pairs, and $4 * 3$ ways to seat them, with another 2^2 ways to exchange boys within pairs, yielding $\binom{4}{2} * 12 * 4 = 288$ seatings for the two pairs of boys. The other 4 can once again sit wherever they wish, giving the total number of ways to sit as $288 * 4!$.

For three pairs of boys retaining original partnerships, there are $\binom{4}{3}$ ways to pick the pairs, and $4 * 3 * 2$ ways to seat them, with another 2^3 ways to exchange boys within pairs, yielding $\binom{4}{3} * 24 * 8 = 768$ ways to seat the three pairs of boys. The other 2 can sit wherever they wish, giving a total of $768 * 2!$.

For all boys retaining their original partnerships, one gets $\binom{4}{4} * (4 * 3 * 2 * 1) * 2^4 = 384$ ways to seat the four pairs. There are no more boys to seat, and thus this is the total number of ways to sit.

By the inclusion-exclusion principle, the total number of ways to seat 8 boys without repeating the partnerships from the prior trip is:

$$40320 - 32 * 720 + 288 * 24 - 768 * 2 + 384 = 23040$$

I've given full credit if you've got it right upto this point. But this overcounts by a factor of 8 because of the translation from a circular permutation into a linear permutation. Thus, the total number of ways to seat the boys around the carousel as required is 2880.

Special Problem 1

Arrange $\{A, E, H, I, M, O, U, Y\}$ without creating HE, ME, YOU.
Total number of ways to arrange letters: 8!
Arrangements containing HE (lump HE as a single letter): 7!
Arrangements containing ME: 7!
Arrangements containing YOU: 6!
Arrangements containing HE, ME (overlapping letter): 0
Arrangements containing HE, YOU: 5!
Arrangements containing ME, YOU: 5!
Arrangements containing HE, ME, YOU (overlapping letter): 0
Total number of ways to arrange letters:

$$8! - 7! - 7! - 6! + 0 + 5! + 5! - 0 = 29760$$

Special problem 2

First of all, note that in this problem we are considering lattice paths from $(1,1)$ to (n,n) and *not from* $(0,0)$ to (n,n)

The random variable X ranges from 1 to $n-1$. Let us find out the probability that $X = n-k$ where $1 \le k \le n-1$. To move $n-k$ steps upward in the last leg of the journey we must reach the point $(n-1,k)$ and then move on to (n,k). The number of ways of doing this is equal to the number of lattice paths from $(1,1)$ to $(n-1,k)$ which in turn is equal to

$$\binom{(n-2)+(k-1)}{n-2} = \binom{n+k-3}{n-2}$$

Clearly, the number of ways in which to move $n-k$ steps rightward in the last leg of the journey is also equal to this same expression. The total number of ways to get to (n,n) from $(1,1)$ is $\binom{2n-2}{n-1}$. Thus,

$$Prob[X = n-k] = \frac{2\binom{n+k-3}{n-2}}{\binom{2n-2}{n-1}}.$$

So, the expected value

$$E(X) = \sum_{k=1}^{n-1}(n-k)Prob[X = n-k] = \frac{2}{\binom{2n-2}{n-1}}\sum_{k=1}^{n-1}\binom{n-k}{1}\binom{n+k-3}{n-2}$$

To evaluate this last summation, we use the combinatorial identity

$$\sum_{k=-\infty}^{\infty}\binom{R-k}{r}\binom{S+k}{s} = \binom{R+S+1}{r+s+1}.$$

And, upon simplifying, we get $E(X) = \frac{2n-2}{n}$.

To find the variance of X we use the identity

$$var(X) = E(X^2) - (E(X))^2.$$

We have already evaluated $E(X)$ and we need to evaluate $E(X^2)$. Since X^2 can only equal one of the values $1^2, 2^2, 3^2, \ldots, (n-1)^2$ and $Prob[X^2 = (n-k)^2] = Prob[X = n-k]$, we have

$$E(X^2) = \sum_{k=1}^{n-1}(n-k)^2 Prob[X = n-k].$$

Now we note that $(n-k)^2 = 2\left(\binom{n-k}{2} + \frac{n-k}{2}\right)$. Therefore our problem reduces to evaluating the summation

$$\sum_{k=1}^{n-1}\binom{n-k}{2}\binom{n+k-3}{n-2}.$$

Which, by the above-mentioned combinatorial identity, is equal to $\binom{2n-2}{n+1}$. Plugging this in, and simplifying, we get

$$\begin{aligned} var(X) = E(X^2) - (E(X))^2 &= \left(\frac{4(n-1)(n-2)}{n(n+1)} + \frac{2(n-1)}{n}\right) - \left(\frac{2(n-1)}{n}\right)^2 \\ &= \frac{2(n-1)^2(n-2)}{n^2(n+1)}. \end{aligned}$$