Solutions to Homework Set 3

Let $a = 2x, \quad b = -y$. We are then looking to expand $(a+b)^7$, which is simply:

$$\binom{7}{0}a^0b^7 + \binom{7}{1}a^1b^6 + \binom{7}{2}a^2b^5 + \binom{7}{3}a^3b^4 + \binom{7}{4}a^4b^3 + \binom{7}{5}a^5b^2 + \binom{7}{6}a^6b^1 + \binom{7}{7}a^7b^0$$

$$= b^7 + 7ab^6 + 21a^2b^5 + 35a^3b^4 + 35a^4b^3 + 21a^5b^2 + 7a^6b + a^7$$

And by substituting a=2x, b=-y, we obtain:

$$= (-y)^7 + 7(2x)(-y)^6 + 21(2x)^2(-y)^5 + 35(2x)^3(-y)^4 + 35(2x)^4(-y)^3 + 21(2x)^5(-y)^2 + 7(2x)^6(-y) + (2x)^7$$

$$= -y^7 + 14xy^6 - 84x^2y^5 + 280x^3y^4 - 560x^4y^3 + 672x^5y^2 - 448x^6y + 128x^7$$

Problem 10:

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{n-1} + \binom{n-3}{k-1}$$

First, let us assume a set S with three distinct elements, a, b, and c, and with a total of n elements ($n \geq 3$).

On the left-hand side of the equal sign, let $\binom{n}{k}$ be the number of ways to select k-combinations of S. Let $\binom{n-3}{k}$ be the number of k-combinations of S not containing a, b, or c. Note that $\binom{n}{k} - \binom{n-3}{k}$ is thus the number of k-combinations of S containing at least one of a, b, or c.

On the other hand, let $\binom{n-1}{k-1}$ be the number of ways to select k-combinations of S containing a. Let $\binom{n-2}{k-1}$ be the number of ways to select k-combinations of S containing b but not a (so as to not overcount those combinations with both a and b). Let $\binom{n-3}{k-1}$ be the number of ways to select k-combinations of S containing c but not a or b (again to not overcount). Note, then, that $\binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$ is the number of ways to select k-combinations of S containing a or b or c. And thus, both sides are combinatorially equivalent.

Problem 19:

There are clearly three basic cases with which we must be concerned: where $k \leq -1$, $k = 0$, and $k \geq 1$.

If $k \leq -1$, $\binom{-r}{k}$ is by definition 0. $\binom{r+k-1}{k}$ is also by definition 0, since in both cases $k < 0$. Thus, in this case the equation is satisfied.

If $k = 0$, $\binom{-r}{k}$ is by definition 1. $\binom{r+k-1}{k}$ is also by definition 1, since in both cases $k = 0$. Moreover, $(-1)^k = (-1)^0 = 1 \Rightarrow (-1)^k\binom{r+k-1}{k} = 1$, and the equation is once again satisfied.

In the final case, $k \geq 1$, we note that

$$\binom{-r}{k} = \frac{(-r)(-r-1)(-r-2)\ldots(-r-k+1)}{k!}$$

And by pulling out -1 from each of the k terms on top, we get

$$\binom{-r}{k} = (-1)^k\frac{(r)(r+1)(r+2)\ldots(r+k-1)}{k!} = (-1)^k\frac{(r+k-1)!}{k!(r-1)!}$$

But we note that this is simply

$$\binom{-r}{k} = (-1)^k\binom{r+k-1}{k}$$

And again the equation is satisfied.

Problem 27:

We start with the chain for $\emptyset$:

$$\emptyset$$

Next we produce the chain for $\{1\}$:

$$\emptyset \subset \{1\}$$

Next we produce the chains for $\{1,2\}$:

$$\emptyset \subset \{1\} \subset \{1,2\}$$
$$\{2\}$$

Next we produce the chains for $\{1,2,3\}$:

$$\emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\}$$
$$\{3\} \subset \{1,3\}$$
$$\{2\} \subset \{2,3\}$$

Next we produce the chains for $\{1,2,3,4\}$:

$$\emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\} \subset \{1,2,3,4\}$$
$$\{4\} \subset \{1,4\} \subset \{1,2,4\}$$
$$\{3\} \subset \{1,3\} \subset \{1,3,4\}$$
$$\{3,4\}$$
$$\{2\} \subset \{2,3\} \subset \{2,3,4\}$$
$$\{2,4\}$$

Finally we produce the chains for $\{1,2,3,4,5\}$:

$$\emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\} \subset \{1,2,3,4\} \subset \{1,2,3,4,5\}$$
$$\{5\} \subset \{1,5\} \subset \{1,2,5\} \subset \{1,2,3,5\}$$
$$\{4\} \subset \{1,4\} \subset \{1,2,4\} \subset \{1,2,4,5\}$$
$$\{4,5\} \subset \{1,4,5\}$$
$$\{3\} \subset \{1,3\} \subset \{1,3,4\} \subset \{1,3,4,5\}$$
$$\{3,5\} \subset \{1,3,5\}$$
$$\{3,4\} \subset \{3,4,5\}$$
$$\{2\} \subset \{2,3\} \subset \{2,3,4\} \subset \{2,3,4,5\}$$
$$\{2,5\} \subset \{2,3,5\}$$
$$\{2,4\} \subset \{2,4,5\}$$

A surprisingly large number of people managed to lose points on this really simple problem. Even without going through this computation so methodically one can get the right answer (just by inspection).

Problem 29:

Note that this problem is simply the question of maximizing the size of a clutter. The talk show host has a set of 10 jokes, and you want to select as many combinations of them as possible (each combination is the set of jokes he tells each night) without having any subsets (all jokes told on one night are told on another night). Thus, the maximum number of nights you can tune in is $\binom{10}{5} = 252$. This follows from Sperner's Theorem.

Special Problem 1

Let us start out by noting that

$$\sum_{0\leq i\leq m} x^i = \frac{1-x^{m+1}}{1-x}$$

Taking the derivative of both sides yields:

$$\sum_{0\leq i\leq m} ix^{i-1} = \frac{mx^{m+1}-(m+1)x^m+1}{(1-x)^2}$$

And, multiplying both sides by x:

$$\sum_{0\leq i\leq m} ix^{i} = \frac{mx^{m+2}-(m+1)x^{m+1}+x}{(1-x)^2}$$

In our specific case, $x=2, \quad m=n-1$ yields the following two equations of relevance:

$$\sum_{0\leq k\leq n-1} 2^k = \frac{1-2^n}{1-2} = 2^n-1$$

$$\sum_{0\leq k\leq n-1} k2^k = \frac{(n-1)2^{n+1}-n2^n+2}{(1-2)^2} = (n-1)2^{n+1}-n2^n+2$$

Finally, we look at the given problem:

$$\sum_{0\leq k\leq n-1} 2^k \frac{\binom{n-1}{k}}{\binom{n}{k}} = \sum_{0\leq k\leq n-1} 2^k \frac{\frac{(n-1)!}{k!(n-k-1)!}}{\frac{n!}{k!(n-k)!}}$$

$$= \sum_{0\leq k\leq n-1} 2^k \frac{n-k}{n}$$

$$= \sum_{0\leq k\leq n-1} 2^k - \frac{1}{n}\sum_{0\leq k\leq n-1} k2^k$$

$$= 2^n - 1 - \frac{1}{n}((n-1)2^{n+1}-n2^n+2)$$

$$= 2^n - 1 - 2^{n+1} + \frac{2^{n+1}}{n} + 2^n - \frac{2}{n}$$

$$= 2^{n+1} - 2^{n+1} - 1 + \frac{2^{n+1}}{n} - \frac{2}{n}$$

$$= \frac{2^{n+1}}{n} - \frac{2}{n} - 1$$

$$= \frac{2^{n+1}-2-n}{n}$$

Special Problem 2

Let $N = s^6$, $1 \leq n \leq N$. If $\lfloor n^{1/6} \rfloor$ divides n, then you win \$5. Otherwise, you lose \$1. Let us start by counting the total number of ways one could win. In order to win, $\lfloor n^{1/6} \rfloor$ must divide n. There are s possibilities for the value of $\lfloor n^{1/6} \rfloor$ $(1, 2, \ldots, s-1, s)$, each with a corresponding range of n's $(1^6 \ldots 2^6 - 1, 2^6 \ldots 3^6 - 1, \ldots, (s-1)^6 \ldots s^6 - 1, s^6)$

Every jth integer in the range $j^4 \leq n \leq (j+1)^4 - 1$ satisfies j dividing n. In each of these ranges, there are $\lceil \frac{(j+1)^6 - 1 - j^6 + 1}{j} \rceil$ divisible integers.

$$\lceil \frac{(j+1)^6 - 1 - j^6 + 1}{j} \rceil = \lceil \frac{j^6 + 6j^5 + 15j^4 + 20j^3 + 15j^2 + 6j + 1 - 1 - j^6 + 1}{j} \rceil$$

$$= \lceil 6j^4 + 15j^3 + 20j^2 + 15j + 6 + \frac{1}{j} \rceil$$

$$= 6j^4 + 15j^3 + 20j^2 + 15j + 7$$

Thus, the total number of winning spots on the roulette wheel is:

$$N_{win} = 1 + \sum_{j=1}^{s-1} (6j^4 + 15j^3 + 20j^2 + 15j + 7)$$

where the 1 comes from the possibility of hitting s^6 precisely, which is not included in the sum because it is the only winning value for its range.

At this point you may choose one of several forms in which to present the answer as a closed-form expression in s. One way is to use $\sum_{j=1}^{s-1} j^2 = \frac{s(s-1)}{6}$ and similar expressions for $\sum_{j=1}^{s-1} j^3$ and $\sum_{j=1}^{s-1} j^4$. Another way is to reduce the above sum to:

$$1 + \sum_{j=1}^{s-1} (144\binom{j}{4} + 306\binom{j}{3} + 214\binom{j}{2} + 56j + 7)$$

$$= 1 + 144\sum_{j=1}^{s-1}\binom{j}{4} + 306\sum_{j=1}^{s-1}\binom{j}{3} + 214\sum_{j=1}^{s-1}\binom{j}{2} + 28s(s-1) + 7(s-1)$$

$$= 144\binom{s}{4} + 306\binom{s}{3} + 214\binom{s}{2} + 28s^2 - 28s + 7s - 7 + 1$$

$$= 144\binom{s}{4} + 306\binom{s}{3} + 214\binom{s}{2} + 28s^2 - 21s - 6$$

But whatever you do, don't forget to note that the probability of a win is N_{win}/s^6, else your solution isn't complete!

Special Problem 3

(Part A)

In order for the series to go into the seventh game, each team must have won precisely 3 games. There are $\binom{6}{3}$ ways for this to happen (choose 3 games for each team to win from the 6 that have been played). The probability of each of these possibilities happening is $(\frac{1}{2})^6$, since the probability of each game going as planned is $\frac{1}{2}$. Thus, the probability that there will be a seventh game in a best-of-seven series is

$$(\frac{1}{2})^6 \binom{6}{3} = 5/16$$

(Part B)

Assuming that n is odd, this problem is essentially the same as part (a). In order for the nth game to take place, the teams must be tied after the $(n-1)$th game, which means that they must each have won $\frac{n-1}{2}$ (note that if n is odd, $n-1$ is even) games. By the logic in (a), we know that there are $\binom{n-1}{\frac{n-1}{2}}$ ways for the teams to be tied, and a probability of $(\frac{1}{2})^{n-1}$ for each possible way. Thus, the probability that there will be an nth game is:

$$(\frac{1}{2})^{n-1} \binom{n-1}{\frac{n-1}{2}} \qquad (n \text{ odd})$$

Importatnt observation: Many of you have been misled by the fact that there are 40 ways for a series to go into the seventh game and 70 ways the series can go altogether. These numbers are quite irrelevant to the problem at hand *since these "70 ways" are not equally likely.* Another way to see why this approach is wrong is that it never uses the given probability information, namely that each side a 50% chance of winning a game. Suppose one side had a 90% chance of winning... the above "argument" still gives a 40/70 probability of the series going into the seventh game.