COS 341, November 19, 1997
Handout Number 17

Extra-Credit Homework Set

This is an extra-credit assignment. If you decide to do it, the due date is **Wednesday, November 26**, and late submissions will not be accepted. Please note that **no collaboration** is allowed for this special assignment. Otherwise it's open-book, open-notes, just like other homework sets.

Problem 1 [20pts] (a)[5 points] Let $B(x), C(x), D(x), E(x)$ be the generating functions for the sequences

$$b_0, b_1, b_2, \cdots, b_n, \cdots$$
$$c_0, c_1, c_2, \cdots, c_n, \cdots$$
$$d_0, d_1, d_2, \cdots, d_n, \cdots$$
$$e_0, e_1, e_2, \cdots, e_n, \cdots.$$

If $B(x) = C(x)D(x)E(x)$, express b_n as a summation of terms involving the other three sequences.
(b)[15 points] Let $A(x) = \sum_{n\geq 1} a_n x^n$, where $a_1 = a_2 = 1$ and for $n \geq 3$,

$$a_n = \sum_{1\leq k\leq n-2} \sum_{k+1\leq m\leq n-1} (n-m)a_k a_{m-k}.$$

Find a closed-form expression for $A(x)$.

Problem 2 [35pts] Let $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4, v_5, w_1, w_2, w_3\}$ and E consists of the edges $\{v_i, v_j\}$, $1 \leq i, j \leq 5$ and $\{v_i, w_j\}, 1 \leq i \leq 3$, $1 \leq j \leq 3$. Answer each of the following questions; justify your answers.

(a) [5 points] What is $\omega(G)$, the size of the largest clique?
(b) [5 points] What is $\alpha(G)$, the size of the largest independent set?
(c) [5 points] What is $\chi(G)$, the chromatic number?
(d) [5 points] Is G a planar graph?
(e) [5 points] Is G an Eulerian graph?
(f) [10 points] Does G contain a Hamiltonian cycle?

Problem 3 [30pts] For any $n > 0$, let G_n be the ladder graph with n rungs as shown below. Let t_n be the number of spanning trees of G_n. Clearly, $t_1 = 1, t_2 = 4$. Let $T(x) = \sum_{n \geq 1} t_n x^n$.

(a)[5 points] Determine the value of t_3.
(b)[5 points] Determine the value of t_4.
(c)[20 points] Derive a closed-form formula for $T(x)$.

Hint: Let s_n be the number of spanning trees of G_n that contain the edge $\{n, 2n\}$. Note $s_1 = 1$, $s_2 = 3$. Find a recurrence relation involving both s_n's and t_n's.