AI for Physics

Discovering governing equations from data by sparse identification of nonlinear dynamical systems (SINDy)

Steven L. Brunton, Joshua L. Proctor, and J. Nathan Kutz

PNAS, 2016

Presented by Alexander Kreibich

„Daß ich erkenne, was die Welt
im Innersten zusammenhält“

“That I may understand whatever
binds the world’s innermost core together”

Goethe, 1808; Kline, 2003

Roadmap

1. Why learn equations from data?

   Motivation and alternatives to SINDy

2. How does SINDy find a model?

   Candidate functions, sparse regression, and the Lorenz demo

3. What limits SINDy, and how can we extend it?

   Hidden coordinates and noisy data

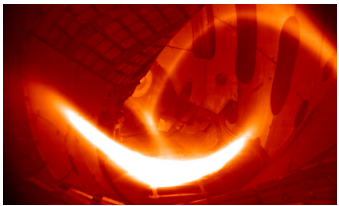
4. Where can we use it?

   Fusion control and discussion

Understanding nature through equations...

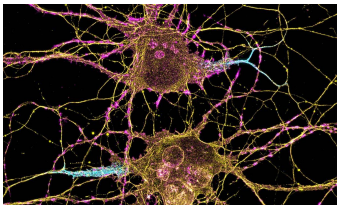
What rules generate the patterns that we observe?

Plasma physics



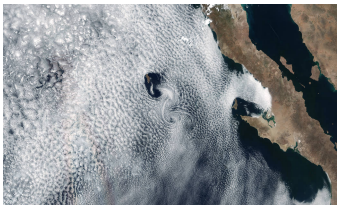
Magnetic confinement

Neuroscience



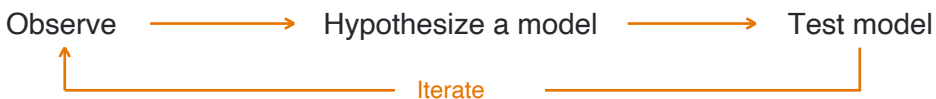
Neuronal signalling

Fluid dynamics



Flow and vortex formation

From Newton onward:



IPP, 2016; Leterrier, 2019; Stevens, 2017

Understanding nature through equations...

Equations describe how patterns evolve

ODEs: a state evolving in time

$$\dot{x} = f(x)$$

Example: interacting neural populations.

PDEs: a field in space and time

$$\partial_t u = F(u, \nabla u, \ldots)$$

Example: fluid velocity or plasma fields.

The relevant equations are often unknown.

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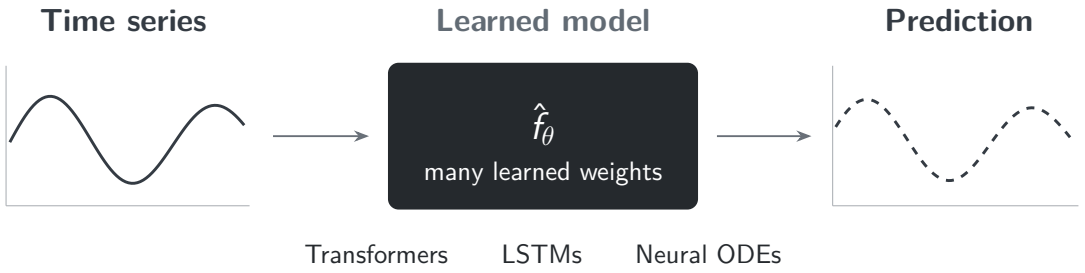
Can we learn the rules from measured data?

Learning a system from data

Neural models can learn useful predictions from measured trajectories.



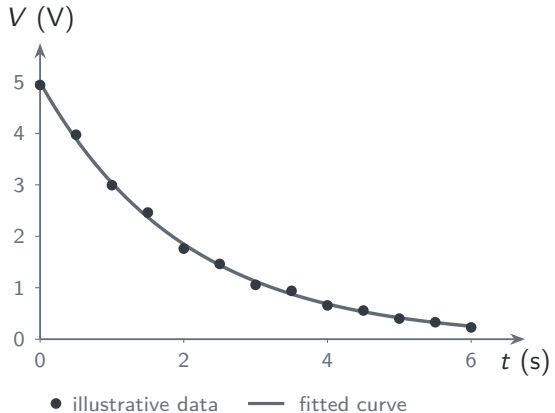
Good prediction does not automatically explain the physics.

The mechanisms can remain hard to read from the model's weights.

Hochreiter, 1997; Vaswani, 2017; Chen, 2018

Before SINDy: parameter regression

Example: the voltage of a discharging capacitor.



Choose the form

$$V(t) = V_0\, e^{-t/\tau}$$

Fit two parameters

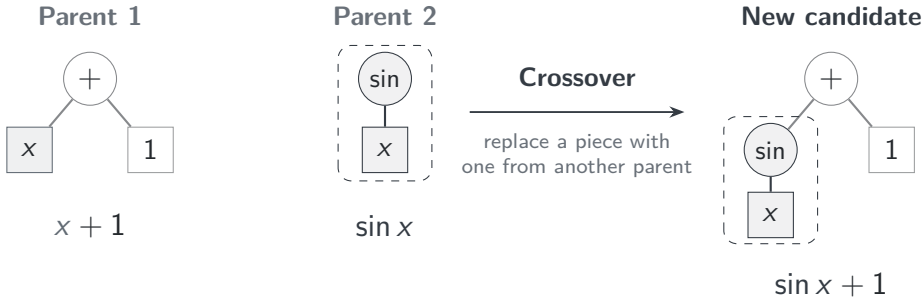
$V_0 \approx 4.99\,\mathrm{V}, \quad \tau \approx 2.01\,\mathrm{s}$

The corresponding ODE is $\dot{V} = -V/\tau$.

Known equation form: learn its parameters from data.

Before SINDy: symbolic regression

Search the equation structure Genetic programming, as used in Eureqa



Mutation

Change one operation: $\sin x + 1 \longrightarrow \cos x + 1$

Test on data $\rightarrow$ keep accurate, simple formulas $\rightarrow$ repeat

SINDy: nonlinear dynamics, linear coefficients

Goal: a symbolic vector field for an autonomous ODE.

$$\dot{x} = f(x) \approx \hat{f}(x) = \Theta(x)\,\Xi$$

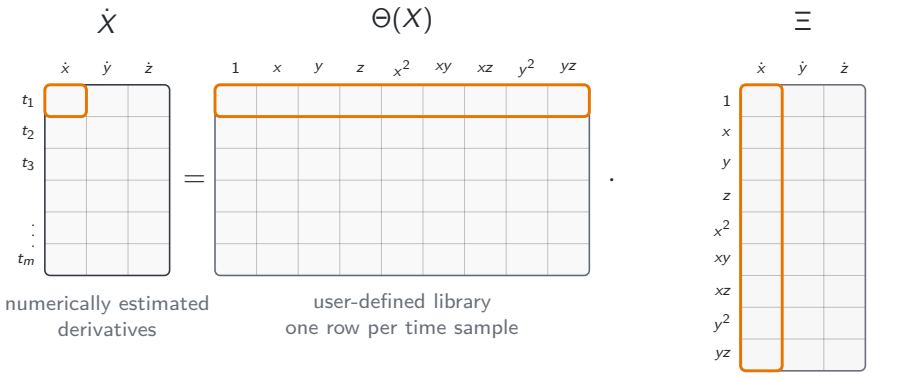
$\Theta(x)$: user-defined candidate functions, possibly nonlinear.

Ξ: learned coefficients, constant in time for this model.

$\dot{x}$: estimated numerically from the measured time series.

$\Rightarrow$ Fix the candidate functions, then fit their coefficients efficiently.

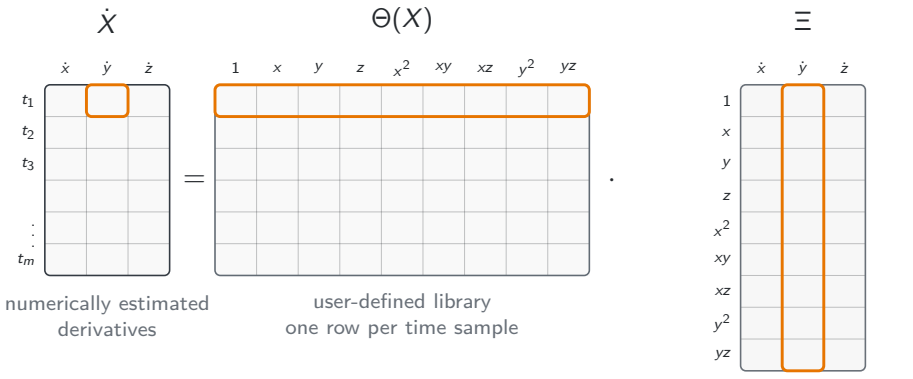
identifying nonlinear dynamics by regression



each entry of $\dot{X}$ is one inner product:
(row of $\Theta(X)$ at that time) · (column of Ξ for that variable)

$$\dot{X} \approx \hat{f}(X) = \Theta(X)\,\Xi$$

identifying nonlinear dynamics by regression



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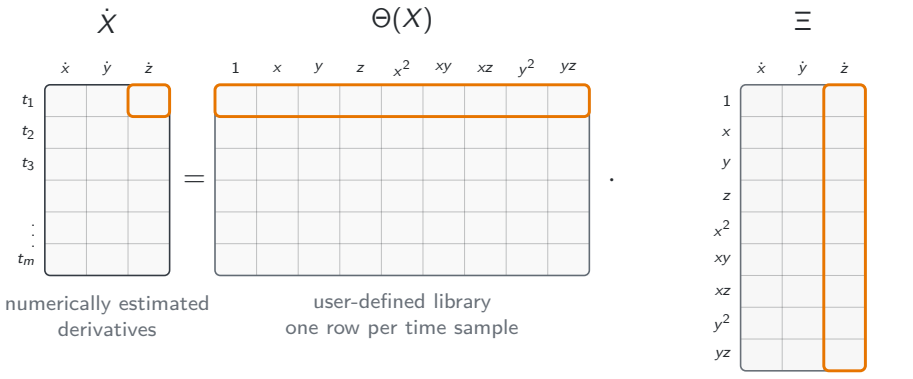

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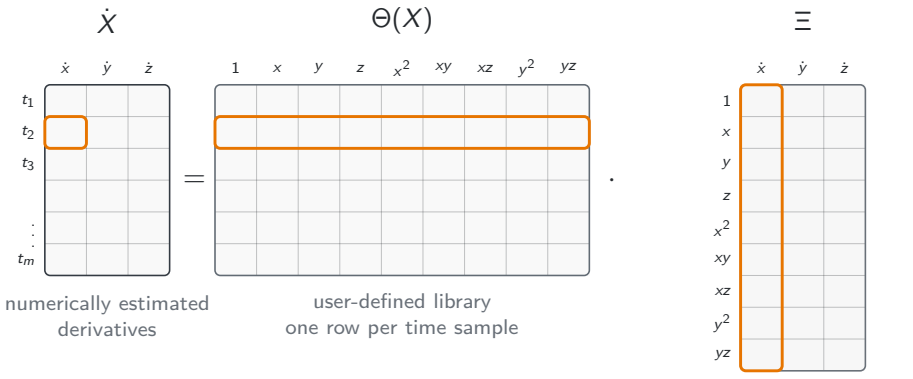

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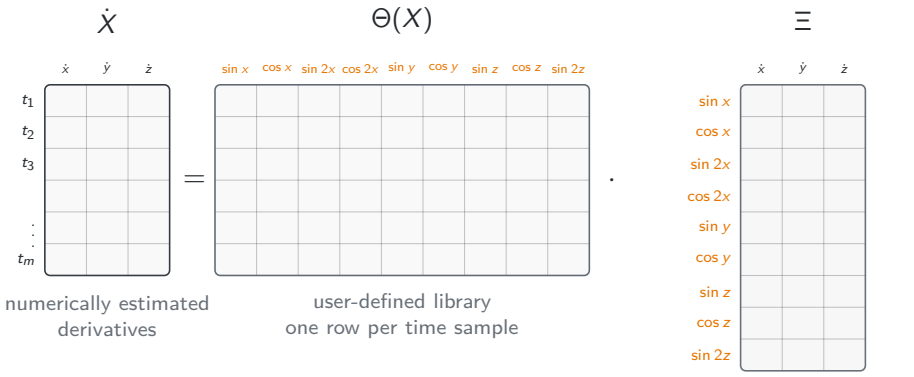
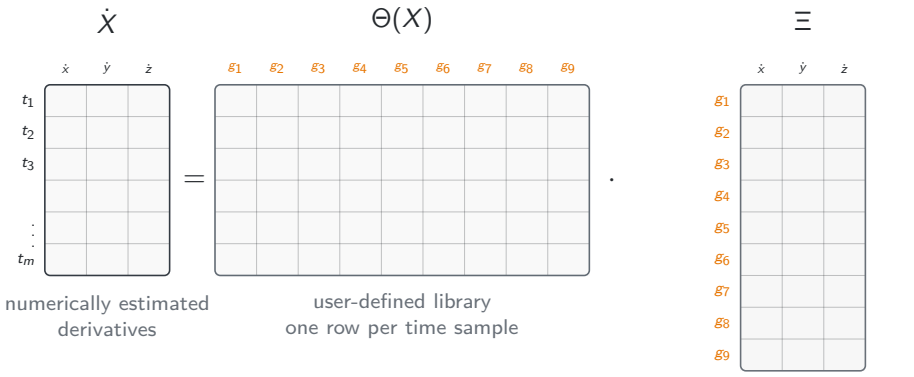
User-defined library: trigonometric functions



Example candidates for oscillatory dynamics; select frequencies using validation.

$$\dot{X} \approx \hat{f}(X) = \Theta(X)\,\Xi$$

User-defined library: Gaussian functions



$g_j(x) = \exp[-\|x - c_j\|^2/(2\sigma_j^2)]$: choose centres and widths.

$$\dot{X} \approx \hat{f}(X) = \Theta(X)\,\Xi$$

Fit the coefficients

Hold the library fixed. Learn one coefficient column per state variable.

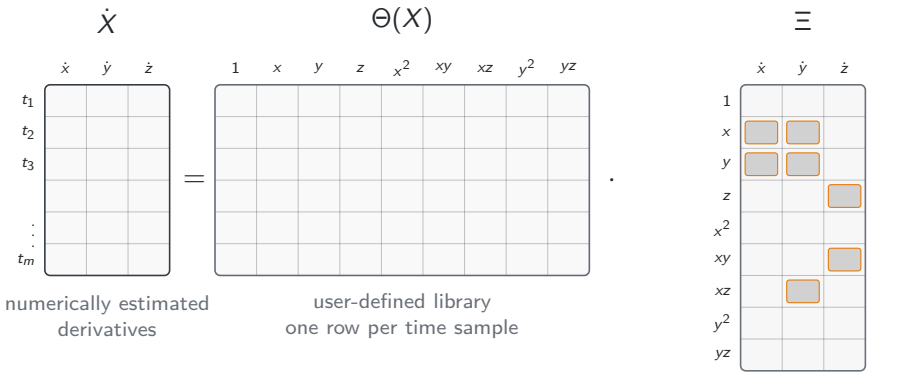
$$\Xi^\star = \arg\min_{\Xi} \|\dot{X} - \Theta(X)\Xi\|_F^2$$

$$\hat{f}(x) = \Theta(x)\Xi^\star$$

Least squares matches the measured rates of change.

It can assign a nonzero coefficient to every candidate term.

SINDy: identify the dynamics by sparse regression

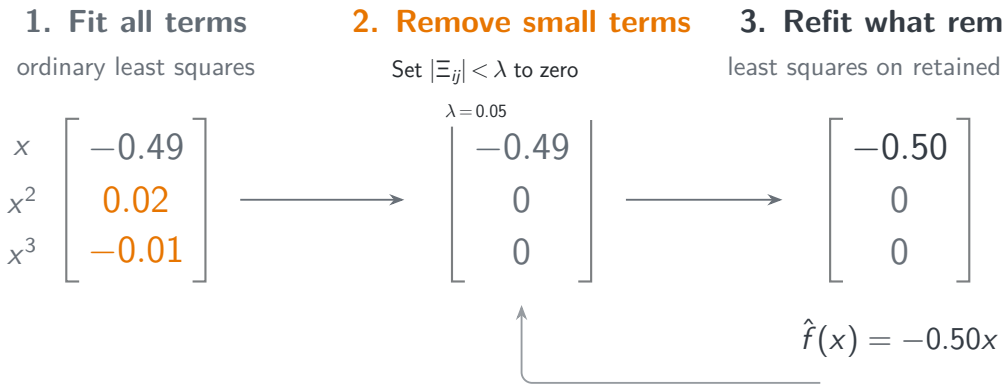


$$\dot{X} \approx \Theta(X)\,\Xi, \qquad \min_{\Xi} \left\| \dot{X} - \Theta(X)\,\Xi \right\|_F^2 + \lambda \|\Xi\|_1$$

Occam's razor: among models that fit well, prefer fewer active terms.

STLSQ: fit, threshold, refit

Sequential thresholded least squares



Repeat thresholding and refitting until the active terms stop changing.

Result 1: Lorenz dynamics

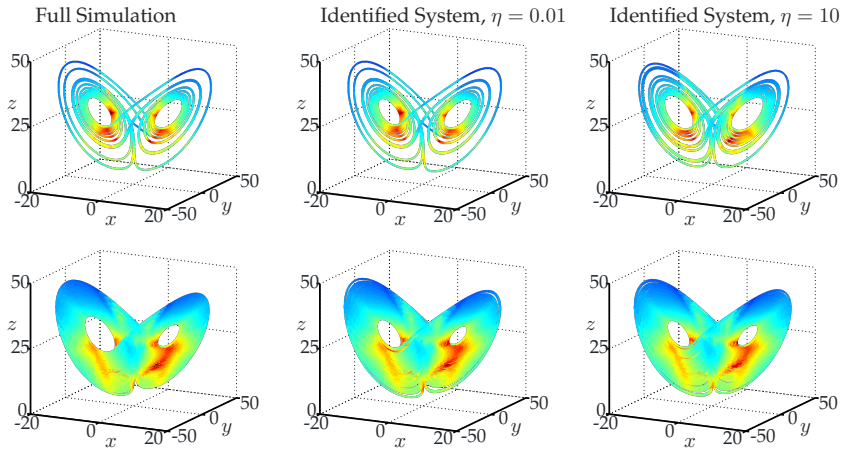
$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x(\rho - z) - y$$

$$\dot{z} = xy - \beta z$$

$$\sigma = 10, \quad \rho = 28$$

$$\beta = 8/3$$



SINDy Recap - Take-home

Data become equations

Learn a dynamical model from measured time series.

Nonlinear dynamics, linear regression

Choose candidate functions and fit their coefficients.

Sparsity keeps the model interpretable

Select a few active terms, then test on new data.

What still does not work well?



What still does not work well?

Choice of coordinates

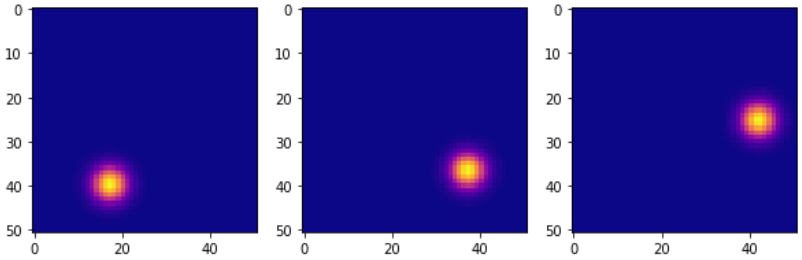
Measured coordinates may not give sparse equations.

Noisy measurements

Finite differences amplify noise.

SINDy-AE: many pixels, simple pendulum motion

We observe images; we want a simple model of the motion



Three snapshots of a synthetic pendulum movie

2,601 pixel values

51 x 51 pixels per frame

→

Few coordinates

Describe the underlying motion

→

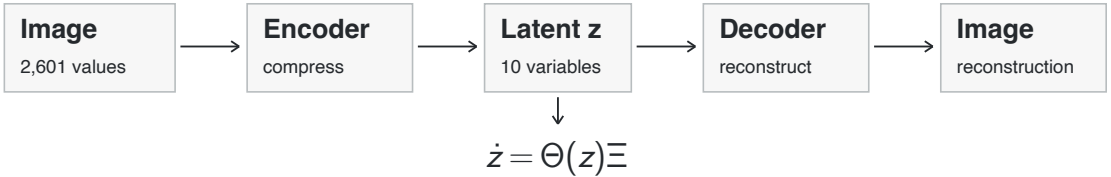
Simple equations

Predict how it evolves

Goal: learn coordinates in which the dynamics become simple

Sillano, 2026; Champion, 2019

SINDy-AE: learn the coordinates and the dynamics



Train both parts together

$$\mathcal{L} = \mathcal{L}_{\text{recon}} + \lambda_x \mathcal{L}_{\dot{x}} + \lambda_z \mathcal{L}_{\dot{z}} + \lambda_s \|\Xi\|_1$$

Reconstruction **Image dynamics** **Latent dynamics** **Sparsity**

Reported result

$$\dot{z}_3 = 0.381 z_8$$

$$\dot{z}_8 = -0.672 z_3$$

Two active latent variables

A simple harmonic oscillator

Small-angle approximation of pendulum motion

Sillano, 2026; Champion, 2019

Why finite differences become fragile

Measurements contain noise: $y_k = x(t_k) + \varepsilon_k$.

$$\dot{x}(t_k) \approx \frac{y_{k+1} - y_k}{h}$$

Differencing subtracts two noisy measurements and divides by h.

$$\mathrm{Var}\left(\frac{\varepsilon_{k+1} - \varepsilon_k}{h}\right) = \frac{2\sigma^2}{h^2}$$

Weak SINDy replaces pointwise differentiation with weighted integrals.

Weak SINDy: move the derivative to a test function

1. Start with standard SINDy

$$\dot{x}(t) = \Theta(x(t))\,\Xi$$

Standard SINDy estimates derivatives from the data.

Weak SINDy: move the derivative to a test function

2. Multiply by a smooth test function

$$\phi(t)\dot{x}(t) = \phi(t)\Theta(x(t))\,\Xi$$

Integrate over a time window:

$$\int_a^b \phi(t)\dot{x}(t)\,dt = \int_a^b \phi(t)\Theta(x(t))\,dt\,\Xi$$

$\phi(t)$ is a smooth weighting function on $[a, b]$.

Weak SINDy: move the derivative to a test function

3. Integrate by parts

$$\int_a^b \phi(t)\dot{x}(t)\,dt = [\phi(t)x(t)]_a^b - \int_a^b \phi'(t)x(t)\,dt$$

$$[\phi x]_a^b = \phi(b)x(b) - \phi(a)x(a)$$

Choose $\phi(a) = \phi(b) = 0$.

The derivative moves from the data to the known test function.

Weak SINDy: move the derivative to a test function

4. The boundary term vanishes

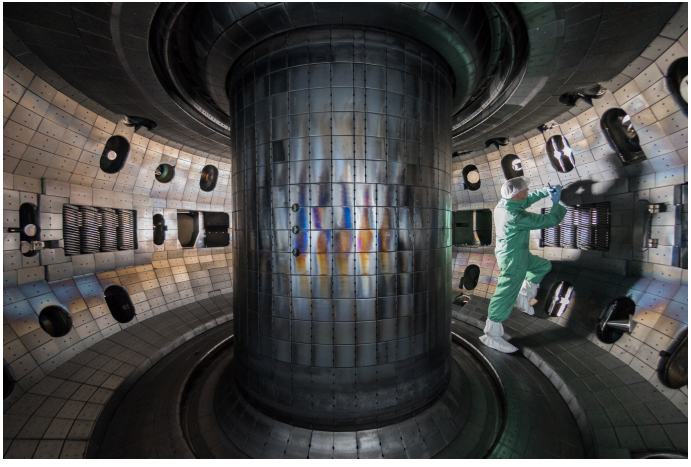
$$[\phi x]_a^b = 0$$

$$-\int_a^b \phi'(t)x(t)\,dt = \int_a^b \phi(t)\Theta(x(t))\,dt\,\Xi$$

$\Longrightarrow$ No numerical $\dot{x}$ needed.

Stack time windows: $B \approx G\Xi$; apply sparse regression.

Fusion: a control problem



Fusion energy

Potential low-carbon energy source

High energy density

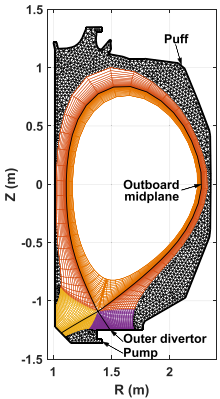
DIII-D
A research tokamak for magnetic confinement

How can we control fusion plasmas?

Rswilcox, 2017; U.S. Department of Energy, 2024

A simulation of the plasma boundary

DIII-D geometry



SOLPS-ITER

B2.5: 2D plasma fluid equations

$$\frac{\partial n}{\partial t} + \nabla \cdot \Gamma_n = S_n \qquad (1a)$$

$$m\frac{\partial n V_\parallel}{\partial t} + \nabla \cdot \Gamma_m = -\nabla p + qnE_\parallel + S_m \qquad (1b)$$

$$\frac{3}{2}\frac{\partial nT}{\partial t} + \nabla \cdot \left(\mathbf{q} + \frac{3}{2}p\mathbf{V}\right) = -p\nabla \cdot \mathbf{V} + Q + S_E \qquad (1c)$$

↕

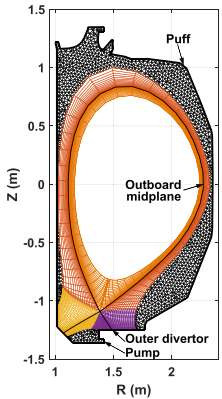
EIRENE

Neutral transport and plasma–wall interactions

How can we control fusion plasmas?

A simulation of the plasma boundary

DIII-D geometry



SOLPS-ITER

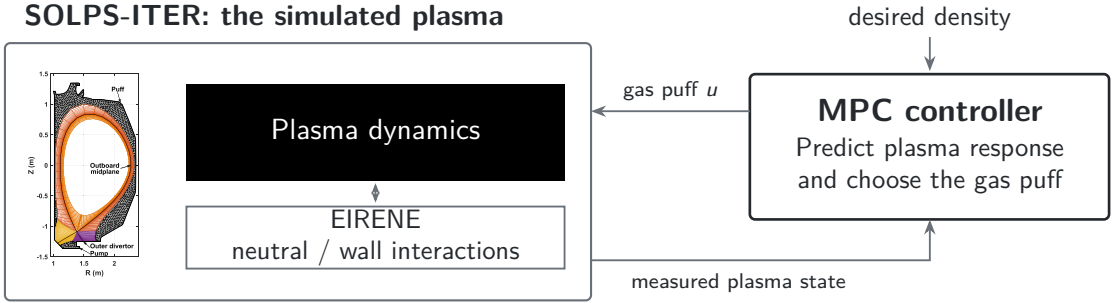
B2.5: 2D plasma fluid equations

Plasma dynamics

EIRENE
Neutral transport and plasma–wall interactions

How can we control fusion plasmas?

MPC needs a model it can evaluate repeatedly

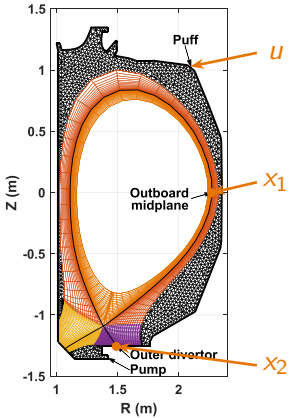


$10\,\mathrm{s}$ **compute** / $10^{-5}\,\mathrm{s}$ **plasma** $\Rightarrow$ ≈ 12 **days** / $1\,\mathrm{s}$

For prediction inside MPC, we need a much simpler model.

How can we control fusion plasmas?

SINDy learns a small model for the controller



$x_1 = n_{e,\mathrm{sep}}^{\mathrm{OMP}}$ electron density
outboard midplane, separatrix

$x_2 = 1/T_{e,\mathrm{sep}}^{\mathrm{div}}$ inverse electron temperature
outer divertor, separatrix

$u = \Gamma_{\mathrm{D}_2}$ deuterium gas puff

Library sketch: constant, linear, quadratic, ...

$\Theta = [1,\ x_1, x_2, u,\ x_1^2, x_1x_2, x_2^2, x_1u, x_2u, u^2, \ldots]$

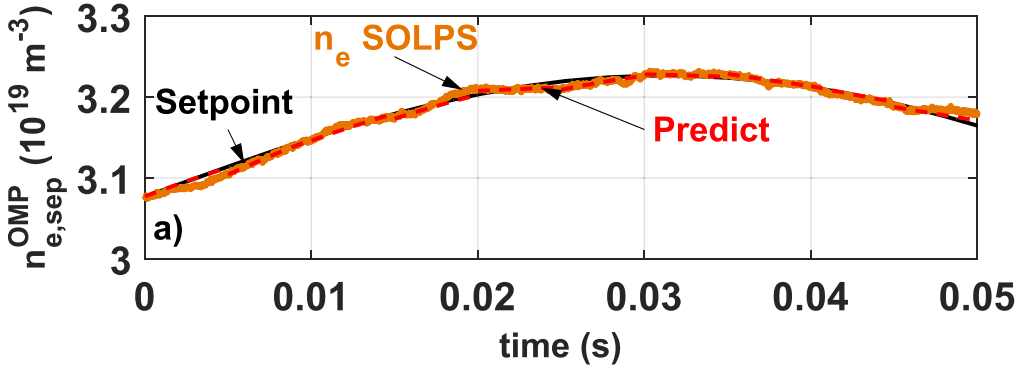
$\dot{x} = \Theta(x, u)\Xi$ Compact ODEs inside MPC

≈ 5 **ms to fit a model from 1 s of simulation data**

How can we control fusion plasmas?

The learned model enables density control

Closed-loop test: SINDy inside MPC, SOLPS-ITER as the plasma



The simulated electron density follows the requested trajectory.

How can we control fusion plasmas?

Summary

1. **From data to symbolic equations**

2. **A simple regression scheme**

3. **Simple equations through sparsity**

   A sparsity-enforcing penalty or threshold selects few terms.

4. **Applications across physics**

   From chaotic systems and fluids to fusion control and climate.

Brunton, 2016; Lore, 2023; Nicoll, 2026

Discussion: recap

1. How do we enforce sparsity in SINDy?

2. What is the issue with fitting models to numerical derivatives?

Discussion: looking beyond the examples

1. What happens when the system is only
   partially observed?

2. When would you choose symbolic equations
   over a neural network?

3. When would SINDy be the wrong approach?
   Which assumptions would fail?

4. How would you balance model accuracy and simplicity,
   and how would you justify the trade-off?

Beyond ODEs: discover PDE terms

So far: learned ODE right-hand side.

What changes when learning a PDE?



Beyond ODEs: discover PDE terms

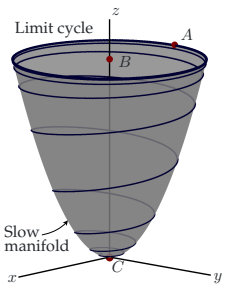
Include spatial derivatives in the candidate library.

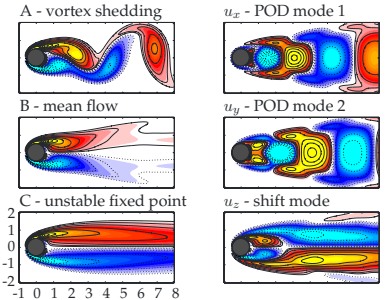
$$\Theta = [1, u, u^2, u_s, u_{ss}, uu_s, \ldots]$$

$$u_t = \Theta\xi \qquad \text{e.g.} \quad u_t = -uu_s + \nu u_{ss}$$

Backup slides

Result 2: cylinder wake in three coordinates





Simulation at Re = 100

292,500 vorticity variables
in the spatial discretization.

POD

Three modal coordinates

Two oscillatory POD modes
+ a shift mode for the mean flow.

$$u(\mathbf{r},t) \approx u_{\text{ref}}(\mathbf{r}) + \sum_{k=1}^{3} a_k(t)\varphi_k(\mathbf{r})$$

POD

$$\longrightarrow \quad \dot{\mathbf{a}} = \hat{\mathbf{f}}(\mathbf{a})$$

SINDy for discrete-time dynamical systems

Learn the next state directly from consecutive measurements.

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$$

Pair each measured state with the next one:

$$X^+ \approx \Theta(X^-)\Xi$$

X^-: current states X^+: next states Ξ: sparse coefficients

No numerical differentiation needed.

Takens: reconstructing hidden state from delays

Only one measured quantity? Use its recent history.

$$\mathbf{z}(t) = [y(t), y(t-\tau), \ldots, y(t-(m-1)\tau)]$$

Delay coordinates can reconstruct the state-space dynamics.

For generic smooth dynamics and observations on a compact d-dimensional manifold, m = 2d + 1 coordinates suffice.

This does not guarantee sparse equations in those coordinates.

Takens, 1981

From observations to climate equations

Ocean

Sea-surface temperature

Atmosphere

Air-pressure variability

Rainfall

North Atlantic rainfall

1950-2022 observations

SINDy identifies three coupled ODEs.

Independent test: 1900-1939

Rainfall terms suggest links between ocean and atmosphere.

Balancing accuracy and model complexity

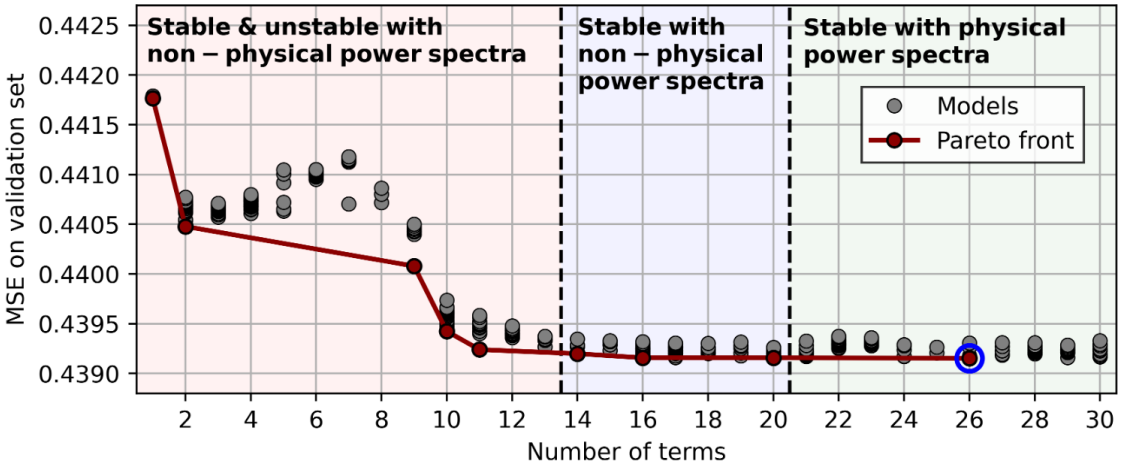
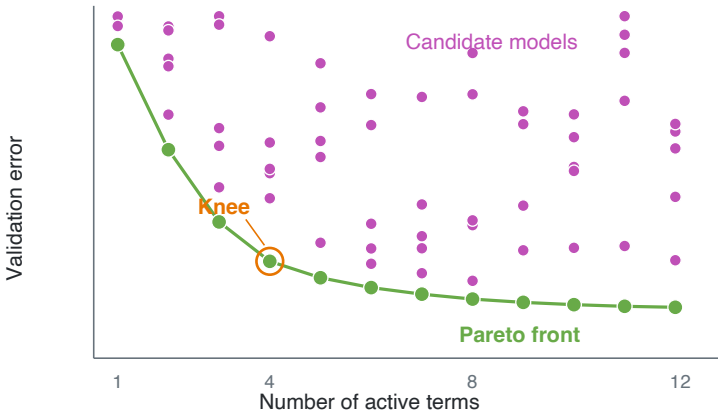


Figure 1: validation error versus model size; Pareto-optimal candidates.

Pareto-optimal models

Balancing prediction error and model complexity



Pareto-optimal

No other candidate is at least as good in both objectives and strictly better in one.

A useful compromise

Near the knee, extra terms bring only small error gains. Choose using held-out data.

Schematic after Kutz (video) and Brunton, 2016

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