

This problem set is due *Wednesday, September 23rd, at 11:59 PM ET*. It will be graded for completion: for every problem, give as much of a solution as you can. If you get stuck, explain where you got stuck, what you are confused about, and what you tried to get around it.

See the AI policy and the collaboration statement in the syllabus. In short, do not use AI. Feel free to collaborate with others, but your write-up must be your own, and you must disclose your collaborators and any outside sources.

Problem 1 (Comparing quantum to classical). Probabilistic systems and quantum systems behave very differently!

- (a) (Reversibility) Quantum operations (except for measurement!) are *reversible*: if I perform an operation taking $|\psi\rangle$ to $U|\psi\rangle$, then there is a matrix $U^\dagger$ which takes $U|\psi\rangle$ back to $|\psi\rangle$. Show that the same is not true for classical operations: give a 2×2 stochastic matrix for which no such inverse exists, and argue why no inverse exists.
- (b) (Continuity) Quantum operations (except for measurement!) can be divided into arbitrarily small pieces: for every unitary matrix U and every positive integer k , there is some unitary matrix V such that $V^k = U$. In other words, V is a “ $1/k$ -fraction” of U , since applying it k times gives back U . This is a nice property to have: actual physical time is (apparently) continuous, so we can be assured that any given unitary can actually exist in this world.¹
 - (i) Show that the same is not true for stochastic matrices:² there is a 2×2 stochastic matrix M such that there is no stochastic matrix N such that $N^2 = M$.
 - (ii) For each of the following two unitary matrices, find a unitary matrix V such that V^2 equals the given matrix. Here, θ is real.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Problem 2 (Dirac notation). Recall the special notation for a two-dimensional state (a qubit); see Section 3.1 of Scott Aaronson’s lecture notes.

- (a) Recall that $|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ and $|-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$. Show that $|i\rangle$ and $|-i\rangle$ are orthonormal, meaning that $\langle i | -i \rangle = 0$ but $\langle i | i \rangle = \langle -i | -i \rangle = 1$.
- (b) Because $|i\rangle$ and $|-i\rangle$ are two orthonormal vectors in two-dimensional space, they form a basis. Write $\frac{i|0\rangle - 3|1\rangle}{\sqrt{10}}$ in that basis.
- (c) Let $|\psi\rangle = \frac{2i|i\rangle + |-i\rangle}{\sqrt{5}}$ and $|\phi\rangle = \frac{2|i\rangle + 3|-i\rangle}{\sqrt{13}}$. What is $\langle \psi | \phi \rangle$? Solve this *without* expanding $|i\rangle$ or $|-i\rangle$ in the $\{|0\rangle, |1\rangle\}$ basis.

¹Note that this property is only true if you allow the V ’s to be complex. This is why complex numbers show up in quantum mechanics, and therefore in our course.

²There are still plenty of stochastic matrices which can be subdivided; these are important for modeling classical many-body systems. For more info, look up “continuous-time Markov chains”.

Problem 3 (Properties of unitary matrices). There are many interesting properties of unitary matrices. So far, I have told you that unitary matrices satisfy $U^\dagger = U^{-1}$, and that this is equivalent to the statement that unitary matrices preserve the norm: $\|U|\psi\rangle\| = \||\psi\rangle\|$. I'll be using more things about unitary matrices, which you will prove now.

- (a) Show that the columns of a unitary matrix U form an orthonormal basis. (You should prove that the columns of U are orthonormal, and that any vector can be written as a linear combination of these columns.)
- (b) Let $|u_1\rangle, \dots, |u_d\rangle$ be one orthonormal basis and let $|v_1\rangle, \dots, |v_d\rangle$ be another one. Show that the matrix X which satisfies $X|u_i\rangle = |v_i\rangle$ for every $i = 1, \dots, d$ is a unitary matrix.

Problem 4 (Rigorously analyzing the quantum Zeno effect). Let $N \geq 1$ be an integer. Consider performing the following algorithm:

- 1. Initialize a qubit to $|0\rangle$;
- 2. For j from 1 to N , measure in the basis

$$\begin{aligned} |\text{GOOD}_j\rangle &= \cos(\pi j/(2N)) |0\rangle + \sin(\pi j/(2N)) |1\rangle; \\ |\text{BAD}_j\rangle &= -\sin(\pi j/(2N)) |0\rangle + \cos(\pi j/(2N)) |1\rangle. \end{aligned}$$

If the outcome is GOOD_j , continue; if the outcome is ever BAD_j , quit running the algorithm.

- (a) If the outcome is GOOD every time, where will the state be at the end of the algorithm?
- (b) For $N = 2$ and $N = 3$, work out explicitly the probability that you never see a BAD outcome.
- (c) For each $k = 1, \dots, N$, show that if I have seen GOOD_j for $j = 1$ through to $j = k - 1$, then the probability I see BAD_k in step k is $\leq (\pi/2)^2/N^2$.
- (d) Then, show that (c) implies that the probability that the algorithm reaches step k and returns BAD_k is $\leq (\pi/2)^2/N^2$, without conditioning on the previous outcomes.
- (e) Finally, use (d) to conclude that the probability of seeing GOOD_j for every $j = 1, \dots, N$ is at least $1 - (\pi/2)^2/N$.