COS 330: Great Ideas in Theoretical Computer Science

Fall 2026

Precept 8

Seeing Double

PROBLEM-SOLVING SESSION

> Today is a **problem-solving session** on LP duality. There are two problems: work on them on your own or in a small group, and ask your TA when you get stuck. Solutions will be shared after precept.

Problem 1: Standard Form

In lecture, we built the dual of an LP of the following form:

$$\begin{aligned} \text{Maximize:} \quad & \sum_j c_j x_j \\ \text{Subject to:} \quad & \sum_j a_{i,j} x_j \leq b_i \quad \forall i \\ & x_j \geq 0 \quad \forall j \end{aligned}$$

We call an LP of this form an LP in *standard form*: it is a maximization problem, every constraint is a "$\leq$" inequality, and every variable is required to be nonnegative.

Each of these three features was used when we built the dual. We multiplied the constraints by nonnegative weights $y_i \geq 0$ and added them up, which is only valid because they are all "$\leq$" inequalities. We then used that the resulting bound $\sum_j \left(\sum_i a_{i,j} y_i\right) x_j$ is at least the objective $\sum_j c_j x_j$ whenever $\sum_i a_{i,j} y_i \geq c_j$ for all j, which is only true because every $x_j \geq 0$. And we wanted an *upper* bound because the objective was being maximized.

Not every LP comes in standard form. However, every LP can be rewritten as an *equivalent* LP in standard form, meaning that there is a correspondence between the feasible solutions of the two LPs that preserves the objective value (possibly up to a sign), so that solving one of them solves the other. In this problem, you will practice this. For each of the following LPs, write an equivalent LP in standard form, and briefly explain why it is equivalent.

(a) Minimization objective

$$\begin{aligned} \text{Minimize:} \quad & 2x_1 + 3x_2 \\ \text{Subject to:} \quad & x_1 + x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(b) Greater-than-or-equal constraints

$$\begin{aligned} \text{Maximize:} \quad & x_1 + 2x_2 \\ \text{Subject to:} \quad & 2x_1 + x_2 \geq 4 \\ & x_1 + 3x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(c) Equality constraints

$$\begin{aligned} \text{Maximize:} \quad & 3x_1 + x_2 \\ \text{Subject to:} \quad & x_1 + x_2 = 5 \\ & 2x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(d) Unconstrained variables

$$\begin{aligned} \text{Maximize:} \quad & 2x_1 + x_2 \\ \text{Subject to:} \quad & x_1 + x_2 \leq 4 \\ & x_1 \geq 0, \quad x_2 \text{ unconstrained (it can be any real number)} \end{aligned}$$

Problem 2: The Dual of the Dual

In lecture, we saw that the dual of the LP (P) below, on the left, is the LP (D) below, on the right:

(P)

$$\begin{aligned} \text{Maximize:} \quad & \sum_j c_j x_j \\ \text{Subject to:} \quad & \sum_j a_{i,j} x_j \leq b_i \quad \forall i \\ & x_j \geq 0 \quad \forall j \end{aligned}$$

(D)

$$\begin{aligned} \text{Minimize:} \quad & \sum_i b_i y_i \\ \text{Subject to:} \quad & \sum_i a_{i,j} y_i \geq c_j \quad \forall j \\ & y_i \geq 0 \quad \forall i \end{aligned}$$

Recall how we built (D): we multiplied each constraint of (P) by a weight $y_i \geq 0$ and added them up, chose the weights so that the result is an *upper* bound on the objective of (P), and then *minimized* that bound. We also claimed in lecture that the dual of the dual is the primal. In this problem, you will prove it.

(a) Since (D) is a minimization problem, its dual should give a *lower* bound on its objective. Build the dual of (D) by following the same steps as in lecture: multiply each constraint of (D) by a nonnegative weight $z_j \geq 0$ and add them up, find conditions on the weights under which the result is a lower bound on the objective $\sum_i b_i y_i$, and then *maximize* that bound. Show that the resulting LP is exactly (P), with z_j in place of x_j.

Hint: It might be tempting to first convert (D) to standard form, as in Problem 1, and then take the dual exactly as in lecture. This works, but it is more complicated than necessary: following the same steps as in lecture directly, with all inequalities flipped, is simpler.