COS 330: Great Ideas in Theoretical Computer Science

Fall 2026

Precept 7

Close Enough

PROBLEM-SOLVING SESSION

Today is a **problem-solving session** on the LP relaxation and rounding framework for approximation algorithms from lecture. There are two problems: work on them on your own or in a small group, and ask your TA when you get stuck. Problem 1 is the main one for today; Problem 2 is a short extension of what we did in lecture, which you can finish at home if you run out of time. Solutions will be shared after precept.

Problem 1: Going in Circles

In class, we saw that rounding the Vertex Cover LP relaxation gives a 2-approximation algorithm. A natural question is whether this analysis is tight: maybe the algorithm actually does better than our proof shows? In this problem, we will find graphs on which LP rounding really does output a vertex cover almost twice as large as the optimal one.

Recall the Vertex Cover problem: given an undirected graph $G = (V, E)$, find a minimum-size subset $S \subseteq V$ such that every edge has at least one endpoint in S. The Integer Program (IP) formulation is:

$$\begin{aligned} \text{Minimize:} \quad & \sum_{v \in V} x_v \\ \text{Subject to:} \quad & x_u + x_v \geq 1 \quad \forall (u, v) \in E \\ & x_v \in \{0, 1\} \quad \forall v \in V \end{aligned}$$

The LP relaxation replaces the integrality constraint with $0 \leq x_v \leq 1$ for all $v \in V$. We write OPT for the size of a minimum vertex cover (the optimal value of the IP), and OPT_{LP} for the optimal value of the LP relaxation. The rounding algorithm from lecture is:

1. Solve the LP relaxation to get an optimal solution x^*_v for all $v \in V$.
2. Return $S = \{v \in V : x^*_v \geq 1/2\}$.

The LP might have several optimal solutions, and the algorithm rounds whichever one the LP solver returns. To know exactly what the algorithm outputs, we will look at graphs whose LP relaxation has a *unique* optimal solution.

For $m \geq 3$, the *cycle* C_m is the graph with vertices $v_1, v_2, \ldots, v_m$, and edges (v_1, v_2), (v_2, v_3), $\ldots$, (v_{m-1}, v_m), and (v_m, v_1). Note that every vertex is an endpoint of exactly two edges.

(a) Consider the triangle C_3. Find OPT, the size of a minimum vertex cover of C_3, and prove your answer.

(b) Show that the LP relaxation for C_3 has a unique optimal solution, namely $x_{v_1} = x_{v_2} = x_{v_3} = 1/2$. What is OPT_{LP}?

(c) What does the rounding algorithm output on C_3? What is the ratio between the size of its output and OPT?

(d) Now consider the odd cycle C_{2k+1}, for any $k \geq 1$. Show that OPT $= k + 1$, but that the rounding algorithm outputs all $2k + 1$ vertices. Conclude that the analysis of LP rounding from lecture is tight: for every $\varepsilon > 0$, there is a graph on which the algorithm outputs a vertex cover of size larger than $(2 - \varepsilon) \cdot$ OPT.

Notice where the factor of 2 is lost. On odd cycles, $\text{OPT}_{\text{LP}} = (2k + 1)/2$ is very close to OPT $= k + 1$, so the LP relaxation itself is an excellent lower bound. The loss comes entirely from rounding: every $x_v^* = 1/2$ gets rounded up to 1, which doubles the value.

Problem 2: Weighted Vertex Cover

In lecture, we saw a 2-approximation algorithm for the unweighted Vertex Cover problem using LP rounding. In this problem, you'll extend the analysis to the weighted version.

Weighted Vertex Cover: Given an undirected graph $G = (V, E)$ where each vertex $v \in V$ has a weight $w_v > 0$, find a vertex cover $S \subseteq V$ that minimizes $\sum_{v \in S} w_v$. Let OPT be the minimum weight of a vertex cover.

The LP relaxation for Weighted Vertex Cover is:

$$\begin{aligned} \text{Minimize:} \quad & \sum_{v \in V} w_v \cdot x_v \\ \text{Subject to:} \quad & x_u + x_v \geq 1 \quad \forall (u, v) \in E \\ & 0 \leq x_v \leq 1 \quad \forall v \in V \end{aligned}$$

The rounding algorithm is the same as in lecture:

1. Solve the LP relaxation to get an optimal solution x_v^* for all $v \in V$.
2. Let $S = \{v \in V : x_v^* \geq 1/2\}$.
3. Return S as the vertex cover.

(a) Prove that S is a valid vertex cover.

(b) Prove that the total weight of S is at most $2 \cdot \text{OPT}_{\text{LP}}$, where OPT_{LP} is the optimal value of the LP relaxation.

(c) Conclude that this algorithm is a 2-approximation for Weighted Vertex Cover.