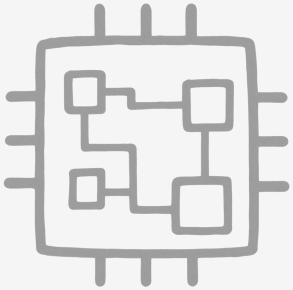


Optimization ▸ Randomized Rounding

Lecture 14

- Background & Motivation
- The Wiring Problem
- LP Relaxation and Rounding
- Randomized Rounding
- Extensions



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Linear Programming in Practice

LPs are used to make real decisions, both directly and as part of larger optimization algorithms.

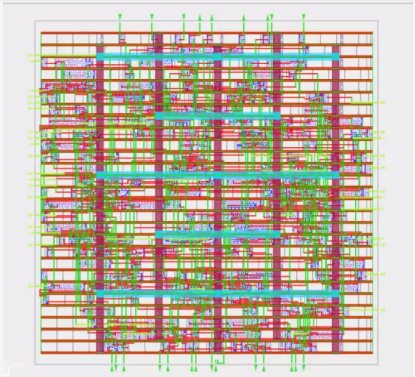
- **Transportation and distribution:** Plan shipments from factories to markets to meet demand at minimum cost.
- **Production planning:** Choose how much of each product to make to maximize profit within material and labor limits.
- **Finance and investment management:** Allocate investments to balance expected returns and downside risk, subject to budget and diversification constraints.

We will focus on another application: **wiring a chip**, where many connections compete for limited routing space.

Wiring a Chip: The Physical Problem

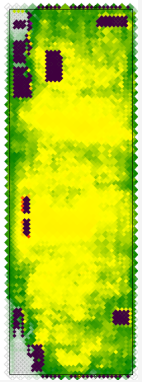
After components are placed, a routing tool must connect their pins using metal wires.

A routed circuit



GCD circuit, OpenROAD example

A routing-congestion map



Industrial microprocessor, fractional routing

Images: OpenROAD tutorial; Blankenburg (2021), Fig. 1(a).

Each connection needs a physical route, but routing space is limited. An LP can help us make these choices.

Recap: LP Relaxation and Rounding

For vertex cover, an LP gave us fractional choices; rounding turned them into an actual cover. We will follow the same recipe for wiring.

Our approximation framework for minimization:

1. Formulate the problem as an integer program.
2. Relax the integrality constraints and solve the LP to obtain a lower bound $\mathrm{OPT}_{\mathrm{LP}} \leq \mathrm{OPT}$.
3. Use the fractional solution to construct a valid integral solution of cost A. This is rounding the LP.

Prove that the rounding loses at most a factor α:

$$A \leq \alpha \, \mathrm{OPT}_{\mathrm{LP}} \leq \alpha \, \mathrm{OPT}.$$

Vertex cover: Choose every vertex whose LP value is at least $1/2$. This deterministic rounding rule gave a factor-two approximation.

Today's Topic: Randomized Rounding

Today we will use randomness in the rounding step.

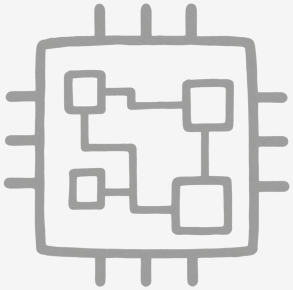
Definition: Randomized rounding

Convert a fractional LP solution into integral choices using randomness. The fractional values determine the probabilities of the choices.

For wiring: Use the fractional route weights as probabilities for choosing one route per connection.

For the analysis: Use concentration inequalities to control congestion across all the channels.

Chip routing was an application in Raghavan and Thompson's original work (1987). A related implementation was tested on industrial gate arrays.



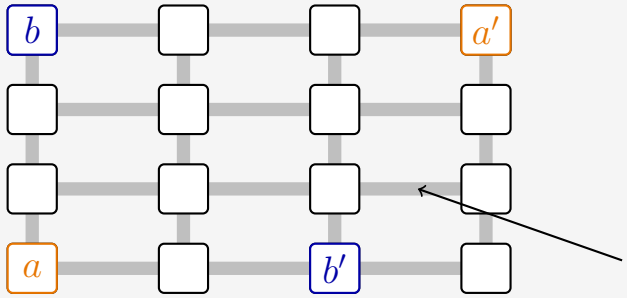
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Wiring: A Simplified Model

For our analysis, replace the chip by an $n \times n$ grid of vertices. Fix the component locations and specify pairs of grid vertices to connect.



Connect a to a'.

Connect b to b'.

Gray corridors are routing channels.

Each gray grid edge is a channel: a corridor that can hold several separate wires. Each wire needs one track in every channel it uses.

We focus on crowded channels, setting aside detailed layer and spacing rules.

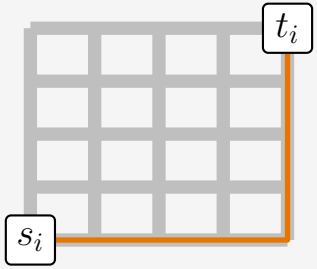
How can we choose routes so that no channel becomes too crowded?

Wiring: The Allowed Routes

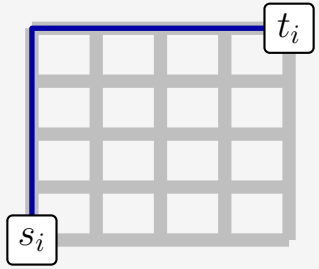
For each connection i, we are given a nonempty list of candidate paths $P_{i,1}, \ldots, P_{i,r_i}$. Paths follow grid edges and do not repeat vertices.

Example: Three candidate paths for the same endpoint pair (s_i, t_i).

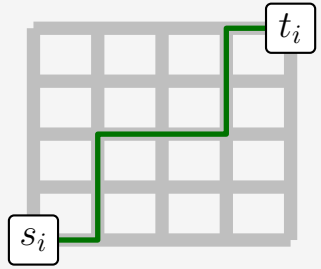
$P_{i,1}$



$P_{i,2}$



$P_{i,3}$



We must choose **one complete path** for each connection. There may be many candidates, with more than one bend.

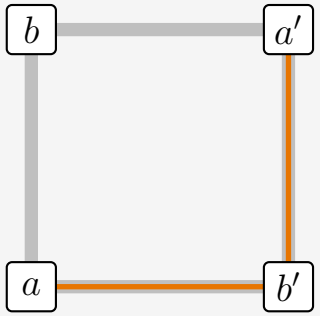
The route lists are part of the input. Our optimization chooses among them.

Example: Two Routes on a Square

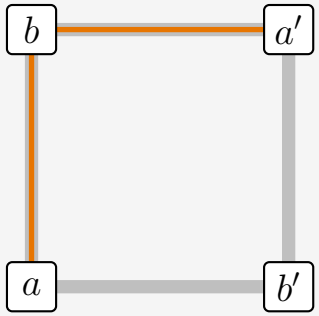
Running example: Four components at the corners of a square.

In this 2×2 grid, each diagonal pair has just two simple paths: horizontal then vertical (H), or vertical then horizontal (V).

Horizontal first: H



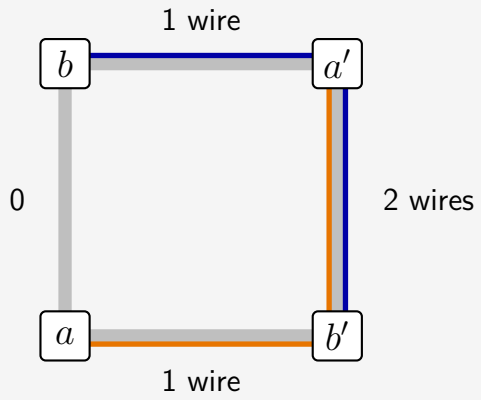
Vertical first: V



The orange wire connects a to a'. Both routes have the same length. We will also need a separate wire connecting b to b'.

This small example has two candidates per connection; the general problem can have many more.

Wiring: Load and Congestion



Definition: Congestion

The load of channel e is the number L_e of wires using it. The congestion of a routing is its largest channel load:

$$\max_e L_e.$$

Both wires here take their H route. The congestion is 2.

This routing needs room for two wires in its busiest channel. Our objective is to make the largest load as small as possible.

Problem: Wiring

Problem

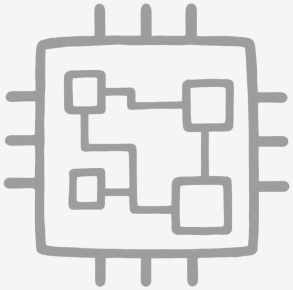
Input: An $n \times n$ grid graph $G = (V, E)$, with $n \geq 2$, and $m \geq 1$ pairs of distinct vertices to connect. For pair i, a nonempty list $P_{i,1}, \ldots, P_{i,r_i}$ of candidate paths is supplied.

Valid solution: Choose one candidate path for each of the m pairs. Each chosen route represents one wire.

Objective: Minimize the congestion, $\max_{e \in E} L_e$.

Notation: Each required pair is a connection. The grid edges are the channels. Their number is $M = |E| = 2n(n-1)$: $n(n-1)$ horizontal and $n(n-1)$ vertical.

Write OPT for the minimum congestion among these valid routings.



Optimization ▸ Randomized Rounding

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Wiring: Decision Variables

Index the m connections by $i \in \{1, \ldots, m\}$. Connection i has candidate paths $P_{i,1}, \ldots, P_{i,r_i}$.

Definition: Route variables

For each candidate $j \in \{1, \ldots, r_i\}$, let

$$x_{i,j} = \begin{cases} 1 & \text{if connection } i \text{ takes path } P_{i,j}, \\ 0 & \text{otherwise.} \end{cases}$$

Exactly one route must be chosen:

$$\sum_{j=1}^{r_i} x_{i,j} = 1.$$

Wiring: The Integer Program

For a general grid, each channel inequality sums the variables for the routes that use that channel.

$$
\begin{aligned}
\text{minimize} \quad & w \\
\text{subject to} \quad & \sum_{j=1}^{r_i} x_{i,j} = 1 && \text{for each connection } i, \\
& \sum_{\substack{i,j \\ e \in P_{i,j}}} x_{i,j} \leq w && \text{for each channel } e, \\
& x_{i,j} \in \{0, 1\}.
\end{aligned}
$$

Each feasible solution selects one complete route per connection. At the optimum, w equals the congestion of the selected routing.

Wiring: The LP Relaxation

Keep the objective and channel constraints. Allow the route variables to take fractional values:

$$\begin{aligned}
&\text{minimize} && w && \\
&\text{subject to} && \sum_{j=1}^{r_i} x_{i,j} = 1 && \text{for each connection } i, \\
& && \sum_{\substack{i,j \\ e \in P_{i,j}}} x_{i,j} \le w && \text{for each channel } e, \\
& && x_{i,j} \ge 0 && \text{(relaxed integrality)}
\end{aligned}$$

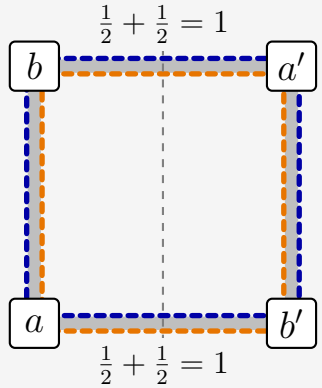
We replaced $x_{i,j} \in \{0, 1\}$ by $x_{i,j} \ge 0$. The sum-to-one constraints already imply $x_{i,j} \le 1$.

We can solve the LP in time polynomial in the supplied route lists' total size.

Write (x^*, C) for an optimal fractional solution and its congestion. As usual, the relaxation gives the lower bound $C \le \mathrm{OPT}$.

LP Relaxation: Fractional Wires

In the square example, split each connection between its two candidate routes.



For each pair, send $1/2$ along H and $1/2$ along V.

Every channel has fractional load 1.

The dashed boundary separates both pairs' endpoints. It must carry total fractional load 2 through the top and bottom channels. One of them has load at least 1, so this solution is optimal.

This solves the fractional problem. Our original problem still needs one complete route.

Algorithm: Deterministic Rounding

Let $r = \max_i r_i$ be the largest number of candidates for a connection.

Algorithm: Choose the largest fraction

For each connection i, choose an index j maximizing $x^*_{i,j}$. Break ties arbitrarily.

Theorem: A factor-r approximation

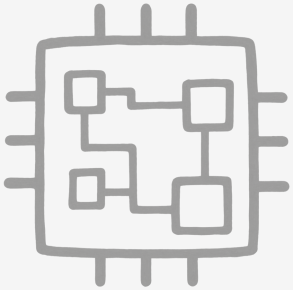
The resulting routing has congestion at most $rC \leq r\mathrm{OPT}$.

Proof

Let $\tilde{x}_{i,j}$ indicate whether connection i chooses route j. The r_i fractions sum to 1, so the largest is at least $1/r_i \geq 1/r$. Therefore $\tilde{x}_{i,j} \leq r x^*_{i,j}$.

For every channel e,

$$L_e = \sum_{\substack{i,j \\ e \in P_{i,j}}} \tilde{x}_{i,j} \leq r \sum_{\substack{i,j \\ e \in P_{i,j}}} x^*_{i,j} \leq rC \leq r\mathrm{OPT}.$$



Optimization ▸ Randomized Rounding

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Algorithm: Randomized Rounding

Recall: A connection i is a required endpoint pair (s_i, t_i). Its candidate paths $P_{i,1}, \ldots, P_{i,r_i}$ all join that same pair. We need one complete wire for each connection.

Algorithm: Randomized rounding

1. Solve the LP to obtain (x^*, C).
2. For each connection i, sample **exactly one** candidate path:

$$\Pr[\text{choose path } P_{i,j}] = x^*_{i,j}, \qquad j = 1, \ldots, r_i.$$

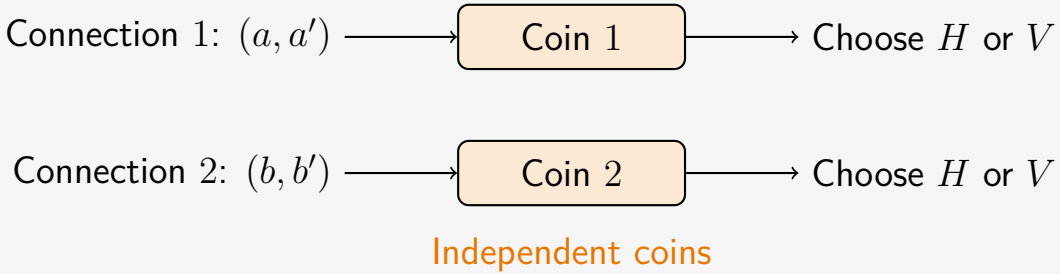
3. Use **independent randomness for different connections**.

Example: Fractions $(0.5, 0.3, 0.2)$ mean selecting one of the three pictured paths with those probabilities, rather than always taking the largest.

The LP values are fixed. The random experiment is choosing one route for each endpoint pair. Different wires may still share channels (grid edges).

Which Choices Are Independent?

A connection is an endpoint pair needing one wire. In the square example, each connection has two candidate paths, H and V, with LP fractions $1/2$ each.



Within a connection: Exactly one path is selected; H and V are mutually exclusive.

Across connections: The path choices are independent, even if the wires share a channel (grid edge).

For a fixed channel, contributions from different connections are independent.

Analysis: The Random Variables

Fix a channel e. Define an indicator for each connection:

$$Y_{i,e} = \begin{cases} 1 & \text{if the chosen route for connection } i \text{ uses } e, \\ 0 & \text{otherwise.} \end{cases}$$

The channel load is $L_e = \sum_{i=1}^{m} Y_{i,e}$, and

$$\Pr[Y_{i,e} = 1] = \sum_{\substack{1 \le j \le r_i \\ e \in P_{i,j}}} x^*_{i,j}.$$

The input and the fractional solution are fixed. Only the choices of complete routes are random.

Analysis: The Channel-Load Lemma

Lemma: Channel load

For every channel e, the load L_e is a sum of independent Bernoulli variables, and its mean satisfies $\mu_e := \mathbb{E}[L_e] \leq C$.

Proof

For a fixed e, each $Y_{i,e}$ depends only on the route chosen for connection i, the endpoint pair (s_i, t_i). Different connections use independent randomness, so $Y_{1,e}, \ldots, Y_{m,e}$ are independent, even when their candidate paths overlap.

By linearity of expectation and the LP constraint for e,

$$\begin{aligned} \mathbb{E}[L_e] &= \sum_i \mathbb{E}[Y_{i,e}] \\ &= \sum_{\substack{i,j \\ e \in P_{i,j}}} x^*_{i,j} \leq C. \end{aligned}$$

Expected Load Is Not Maximum Load

Return to the square, with fair independent route choices for the two pairs.

- Every channel has expected load 1.
- Each path for the first pair shares a channel with either path for the second pair, so every realized routing has maximum load 2.

$$\max_e \mathbb{E}[L_e] = 1, \qquad \mathbb{E}\Big[\max_e L_e\Big] = 2.$$

A small expected load for each channel does not ensure that all channels have small loads in the same routing.

We need to bound the probability of an overload, then control **all M channels at once**.

Analysis: Approximation Guarantee

Theorem: Randomized rounding with small relative error

Let $0 < \varepsilon \leq 1$. If the fractional optimum satisfies

$$C \geq \frac{3 \ln(10M)}{\varepsilon^2},$$

then, with probability at least 0.9, randomized rounding has congestion at most

$$(1+\varepsilon)C \leq (1+\varepsilon)\mathrm{OPT}.$$

The input is fixed. Probability is over the independent route choices.

The square has $C = 1$, so it does not satisfy the hypothesis on C. We will combine channel concentration with a union bound to prove the theorem.

Analysis: The Chernoff Lemma

Recall from Lecture 6: For a sum X of independent Bernoulli variables, with mean $\mu > 0$,

$$\Pr[X \geq \mu + s] \leq \exp\left(-\frac{s^2}{2\mu + s}\right) \qquad (s > 0).$$

We know $\mu_e \leq C$, but channels need not all have mean exactly C.

Lemma: Chernoff with a mean upper bound

If X is a sum of independent Bernoulli variables and $\mathbb{E}[X] \leq C$, then for every $t > 0$,

$$\Pr[X \geq C + t] \leq \exp\left(-\frac{t^2}{2C + t}\right).$$

Analysis: One-Channel Concentration

Lemma: Concentration for one channel

For each channel e and $0 < \varepsilon \leq 1$,

$$\Pr[L_e \geq (1+\varepsilon)C] \leq e^{-\varepsilon^2 C/3}.$$

Proof

The channel-load lemma gives independent Bernoulli contributions with mean at most C. Apply the Chernoff lemma with $t = \varepsilon C$:

$$\begin{aligned}\Pr[L_e \geq (1+\varepsilon)C] &\leq \exp\left(-\frac{\varepsilon^2 C^2}{2C + \varepsilon C}\right) \\ &= e^{-\varepsilon^2 C/(2+\varepsilon)} \leq e^{-\varepsilon^2 C/3}.\end{aligned}$$

The last step uses $\varepsilon \leq 1$.

Analysis: Simultaneous Concentration

Recall that $M = |E| = 2n(n-1)$ is the number of channels.

Lemma: Concentration for all channels

For $0 < \varepsilon \leq 1$,

$$\Pr\Big[\max_e L_e > (1+\varepsilon)C\Big] \leq Me^{-\varepsilon^2 C/3}.$$

Proof

A bad routing has at least one channel with load greater than $(1+\varepsilon)C$. By the union bound and the one-channel concentration bound,

$$\begin{aligned}\Pr\Big[\max_e L_e > (1+\varepsilon)C\Big] &\leq \sum_{e\in E} \Pr[L_e > (1+\varepsilon)C] \\ &\leq M\exp(-\varepsilon^2 C/3).\end{aligned}$$

The Approximation Guarantee

When $C \geq 3\ln(10M)/\varepsilon^2$, the probability of any overload is at most

$$\Pr\Big[\max_e L_e > (1+\varepsilon)C\Big] \leq M\exp(-\varepsilon^2 C/3) \leq M\exp(-\ln(10M)) = \frac{1}{10}.$$

With probability at least 0.9, **all channels simultaneously** satisfy

$$L_e \leq (1+\varepsilon)C \leq (1+\varepsilon)\mathrm{OPT}.$$

Why the Fractional Optimum Matters

For $0 < \varepsilon \leq 1$, our failure bound was

$$M \exp(-\varepsilon^2 C/3).$$

- A fixed relative overload becomes exponentially less likely as C grows.
- Controlling M channels costs a factor of M in probability.
- An exponent of order $\ln M$ pays for that factor.

The rounding error can be small **relative to the load**, even though every wire is indivisible.

There is no extra factor for the number of candidate paths: a connection contributes at most one wire to a fixed channel.

Can we check whether a rounded routing meets our target?

The LP Also Lets Us Check the Answer

After rounding, compute the actual congestion A.

For every nonempty instance,

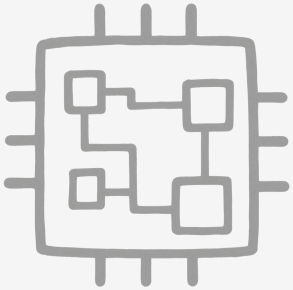
$$C \le \mathrm{OPT} \le A, \qquad \frac{A}{\mathrm{OPT}} \le \frac{A}{C}.$$

So A/C is a certificate of the returned routing's approximation factor.

Algorithm: Repeat the rounding

Solve the LP once. If C satisfies the theorem's hypothesis, generate independent rounded routings until one has congestion at most $(1+\varepsilon)C$. Check the congestion after each attempt.

Each attempt succeeds with probability at least 0.9. By independence, the probability that all k attempts fail is at most 10^{-k}.



Optimization ▸ Randomized Rounding

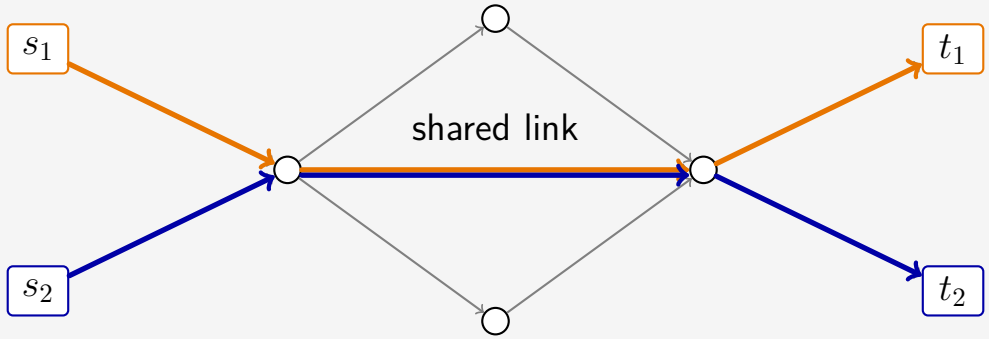
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Extension: Routing on General Graphs

Replace the grid by a general directed graph $G = (V, E)$. Vertices are network locations; edges are links with limited capacity.

Given source–destination pairs (s_i, t_i), choose a path for each pair. The paths share the capacity of the network links.



How can we share link capacity across many pairs?

Extension: Multicommodity Flow

One flow for each source–destination pair.

Definition: Multicommodity flow

Given a directed network $G = (V, E)$ with capacities u_e and pairs (s_i, t_i), each pair is a commodity. For unit demands:

- Send one unit of flow $f_i \geq 0$ from s_i to t_i, allowing splitting.
- Share edge capacities:

$$\sum_i f_i(e) \leq u_e \qquad \text{for every edge } e.$$

Flow conservation and capacity constraints are linear: this is an LP.

Extension: Low-Congestion Routing

Choose unit flows f_i to minimize the busiest edge's total flow:

$$\text{minimize } w \qquad \text{subject to } \sum_i f_i(e) \le w \quad \text{for every } e.$$

Let C be the fractional optimum.

Algorithm: Rounding a multicommodity flow

1. Discard cycles; decompose each unit flow into paths with weights summing to 1.
2. Independently choose one path per pair using those weights as probabilities.

Again, $\mathbb{E}[L_e] \le C$. The same Chernoff and union-bound analysis applies, with $M = |E|$.