

Name:
NetID:

COS 302 – Midterm Practice Questions

Problem 1 (??pts)

Ryan's kids, Dorian and Eleanor, love sushi. Like all little kids, however, they really want to make sure things are fair and that neither of them gets to have more than the other one. Ryan orders six pieces of salmon sashimi and twelve Alaska rolls. Ellie and Dorian eat it all up, each eating an equal number of rolls. Can you figure out how many rolls of each type were eaten by each child? Let s_e and s_d be the number of salmon pieces eaten by Eleanor and Dorian, respectively, and let a_e and a_d be the respective number of Alaska rolls they ate.

- (A) Write down a set of equations describing this problem.
- (B) Is it possible to uniquely determine who ate what? Support your argument.
- (C) Dorian complains "*Ellie got twice as many pieces of salmon as me!*". Is the system now *underdetermined* (too few independent constraints: still infinitely many solutions), *overdetermined* (conflicting constraints: no solution), or uniquely solvable? Explain your answer.

Problem 2 (??pts)

What is the range of values for a in the matrix below for which it is positive definite?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

Problem 3 (??pts)

Let $\mathbf{A}$ and $\mathbf{B}$ be positive definite $n \times n$ matrices, and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ where $\mathbf{x} \neq 0$ and $\mathbf{y} \neq 0$. Are the following statements true or false? Circle your answer.

- (A) TRUE — FALSE $\mathbf{A} + \mathbf{B}$ is positive definite.
- (B) TRUE — FALSE $\mathbf{x}^\top \mathbf{A} \mathbf{x} > 0$
- (C) TRUE — FALSE $\det(\mathbf{A}) = 0$
- (D) TRUE — FALSE $\mathbf{A}^\top \mathbf{A}$ is positive definite.
- (E) TRUE — FALSE $\mathbf{A}$ is singular.
- (F) TRUE — FALSE The eigenvalues of $\mathbf{B}$ are negative.

Problem 4 (??pts)

Match the statements.

- (A) A matrix that equals its transpose.
 - (B) A square symmetric matrix $A \in \mathbb{R}^{n \times n}$ such that $\forall \mathbf{x} \in \mathbb{R}^n, \mathbf{x}^\top A \mathbf{x} \geq 0$.
 - (C) A square matrix that has no inverse.
 - (D) For a linear map represented with matrix A , the set of vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$.
 - (E) For a linear map represented with matrix A , the set of vectors $\mathbf{y}$ such that there exists an $\mathbf{x}$ where $A\mathbf{x} = \mathbf{y}$.
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(i) Positive semi-definite matrix
MATCH:

(ii) Image
MATCH:

(iii) Symmetric matrix
MATCH:

(iv) Kernel/nullspace
MATCH:

(v) Singular matrix
MATCH:

Problem 5 (??pts)

Consider the following code snippet and match each variable with its value.

```
import numpy as np

def create_matrix(N):
    vec = np.arange(N) + 1
    return vec + vec.reshape(N, 1)

A = create_matrix(3)
B = A - np.mean(A, axis=0)
C = np.matmul(B, A[:, 2].reshape(3, 1))
D = np.sum(C)
```

(i) 0.0

variable name:

(ii) $\begin{bmatrix} -15. \\ 0. \\ 15. \end{bmatrix}$

variable name:

(iii) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

variable name:

(iv) $\begin{bmatrix} -1. & -1. & -1. \\ 0. & 0. & 0. \\ 1. & 1. & 1. \end{bmatrix}$

variable name:

Problem 6 (??pts)

Consider the vector space $\mathbb{R}^3$.

- (A) Find the projection matrix $\mathbf{P}_\pi$ onto the line through the origin spanned by $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.
- (B) Find the projection $\mathbf{z}$ of the vector $\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ onto the subspace spanned by $\mathbf{b}$.
- (C) Now use the result of (b) to find the projection of $\mathbf{z}$ onto the subspace spanned by $\mathbf{b}$.
- (D) What is the rank of $\mathbf{P}_\pi$?
- (E) Write down an eigenvector with corresponding eigenvalue of $\mathbf{P}_\pi$.

You may leave your answers in terms of unreduced fractions.

Problem 7 (??pts)

Which of the following are valid bilinear maps for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$? Explain your reasoning.

(A) $f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top \mathbf{y} + 1$

(B) $f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top \mathbf{y}$

(C) $f(\mathbf{x}, \mathbf{y}) = \text{tr}(\mathbf{x}\mathbf{y}^\top \mathbf{A})$ for square symmetric $\mathbf{A}$

(D) $f(\mathbf{x}, \mathbf{y}) = \mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top$

Problem 8 (??pts) (A) The matrix that rotates vectors in $\mathbb{R}^2$ counterclockwise by an angle θ is

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

Write down $\mathbf{R} = \mathbf{R}(45^\circ)$ explicitly and apply it to

$$\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

- (B) Without computing anything: what is $\|\mathbf{R}\mathbf{x}\|$ compared to $\|\mathbf{x}\|$? Explain.
- (C) What is $\det(\mathbf{R})$? Answer from the geometric meaning of the determinant, not by the 2×2 formula.
- (D) Is $\mathbf{R}$ an orthogonal matrix? What is $\mathbf{R}^{-1}$, described in words?

Problem 9 (??pts)

Let

$$A = \begin{bmatrix} 9 & 1 \\ 5 & 5 \end{bmatrix}.$$

- (A) Compute the eigenvalues of A .
- (B) Find the eigenvectors of A for each of the eigenvalues you found in part (A).
- (C) Let $\mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Calculate $A^n \mathbf{u}_0$, where $n \geq 0$ is an integer.

Problem 10 (??pts)

Let

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \mathbf{d} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad U = \text{span}[\mathbf{a}, \mathbf{b}] \quad V = \text{span}[\mathbf{a}, \mathbf{c}] \quad \mathbf{A} = [\mathbf{a}, \mathbf{b}, \mathbf{c}]$$

Are the following statements true or false? Please circle TRUE or FALSE for each statement.

- (A) TRUE — FALSE: $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.
- (B) TRUE — FALSE: $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$ are linearly independent.
- (C) TRUE — FALSE: U and $\mathbf{c}$ are orthogonal.
- (D) TRUE — FALSE: V and $\mathbf{b}$ are orthogonal.
- (E) TRUE — FALSE: The dimensions of both $\mathbf{a}$ and $\mathbf{c}$ are 3, so the dimension of V is 3.
- (F) TRUE — FALSE: $\mathbf{A}$ is full rank.
- (G) TRUE — FALSE: $\det(\mathbf{A}) = 0$
- (H) TRUE — FALSE: If we perform elementary transformations on $\mathbf{A}$ until it's in Reduced Row-Echelon Form, then we will definitely obtain an identity matrix in the end.
- (I) TRUE — FALSE: If we perform elementary transformations on $\mathbf{A}$ until it's in Row-Echelon Form, then for the values at the four corners of the derived matrix, there is only one value that we are not sure whether it is zero or non-zero.
- (J) TRUE — FALSE: $\mathbf{d} \in U$
- (K) TRUE — FALSE: $\mathbf{A} + \mathbf{A}^T$ is a square matrix as well as a symmetric matrix.

Problem 11 (??pts)

Let $V = \{(a, b, c) \in \mathbb{R}^3 \text{ such that } a = b^3\}$. Determine whether V is a subspace of $\mathbb{R}^3$.

Problem 12 (??pts)

Let A be an $m \times m$ matrix where $\text{tr}(A) = 4$. The invertible matrix B is also $m \times m$ with $\text{tr}(B) = 7$. I is the $m \times m$ identity matrix.

- (A) What is $\text{tr}(A + B)$?
- (B) What is $\text{tr}(BAB^{-1})$?
- (C) What is $\text{tr}(I + B)$?

Problem 13 (??pts)

The vector $\mathbf{z} \in \mathbb{R}^n$ (where $n > 1$) is unit length, i.e., $\|\mathbf{z}\|_2 = 1$.

- (A) What is $\text{tr}(\mathbf{z}\mathbf{z}^\top)$?
- (B) What is $\det(\mathbf{z}\mathbf{z}^\top)$?
- (C) $\mathbf{z}$ is an eigenvector of the square $n \times n$ matrix $\mathbf{B}$ with eigenvalue λ . What is $\text{tr}(\mathbf{B}\mathbf{z}\mathbf{z}^\top)$?

Problem 14 (??pts)

A 3×3 linear system $A\mathbf{x} = \mathbf{b}$ is known to have the two distinct solutions

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

You are not told what A or $\mathbf{b}$ is.

- (A) Find a nonzero solution of the homogeneous system $A\mathbf{h} = \mathbf{0}$.
- (B) Write down two more solutions of $A\mathbf{x} = \mathbf{b}$, different from $\mathbf{x}_1$ and $\mathbf{x}_2$.
- (C) What is $\det(A)$? Explain how you know.

Problem 15 (??pts)

Let A be an $n \times n$ matrix. For each statement below, answer YES or NO: is it *equivalent* to the statement “ A is invertible”? Give a one-sentence justification for each.

- (A) $\det(A) \neq 0$.
- (B) The columns of A are linearly independent.
- (C) The null space of A is $\{\mathbf{0}\}$.
- (D) A is symmetric.
- (E) Zero is not an eigenvalue of A .
- (F) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbb{R}^n$.

Problem 16 (??pts)

The matrix $\mathbf{B}$ is symmetric, and you are told that $[2, 1]^\top$ is an eigenvector of $\mathbf{B}$ with eigenvalue 5.

- (A) Find an eigenvector of $\mathbf{B}$ that is not a multiple of $[2, 1]^\top$, using no matrix computation.
- (B) The other eigenvalue of $\mathbf{B}$ is -2 . What are $\text{tr}(\mathbf{B})$ and $\det(\mathbf{B})$?
- (C) Is $\mathbf{B}$ positive definite? Positive semidefinite?

Problem 17 (??pts)

Consider the basis for $\mathbb{R}^2$ given by

$$\tilde{\mathbf{b}}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \tilde{\mathbf{b}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

written in standard coordinates, and let $\mathbf{S} = [\tilde{\mathbf{b}}_1 \ \tilde{\mathbf{b}}_2]$.

- (A) The vector $\mathbf{x}$ has standard coordinates $[3, 4]^T$. What are its coordinates in the basis $\{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$? Verify your answer by forming the corresponding linear combination.
- (B) The vector $\mathbf{y}$ has coordinates $[1, 1]^T$ in the basis $\{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$. What are its standard coordinates?
- (C) The linear map Φ has matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ in the standard basis. Compute its matrix $\tilde{\mathbf{A}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ in the basis $\{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$, and check that the trace and determinant came out unchanged.

Problem 18 (??pts)

You observe the data points (t_i, y_i) : $(0, 0)$, $(1, 1)$, $(2, 1)$, and want to fit a line $y \approx w_0 + w_1 t$ by least squares.

- (A) Write down the matrix X and vector $\mathbf{y}$ so that the fit minimizes $\|\mathbf{y} - X\mathbf{w}\|^2$, and explain in one sentence why no exact solution to $X\mathbf{w} = \mathbf{y}$ exists.
- (B) Form and solve the normal equations to find w_0 and w_1 .
- (C) Compute the residual vector and verify that it is orthogonal to both columns of X . Why must that happen?

Problem 19 (??pts)

Three web pages link to each other: A links to B ; B links to A and C ; C links to A .

- (A) A matrix is *column-stochastic* if its entries are nonnegative and each column sums to one. Write down the column-stochastic matrix $\mathbf{M}$ of the random surfer for this graph: entry M_{ij} is the *fraction* of the surfers at page j who move next to page i , with each page splitting its surfers evenly over its out-links. (Order the pages A, B, C .)
- (B) Verify that $\mathbf{p} = \frac{1}{5}[2, 2, 1]^T$ satisfies $\mathbf{M}\mathbf{p} = \mathbf{p}$.
- (C) In a sentence or two: why does repeatedly multiplying a starting vector of surfer fractions by $\mathbf{M}$ converge to (a multiple of) $\mathbf{p}$, and what is this algorithm called?
- (D) Your friend worries about an unlucky start: what if the initial vector contains *none* of $\mathbf{p}$ – say, it is proportional to one of $\mathbf{M}$'s *other* eigenvectors? Show this cannot happen: prove that any eigenvector $\mathbf{v}$ of $\mathbf{M}$ with eigenvalue $\lambda \neq 1$ must have entries that sum to *zero* – whereas a vector of surfer fractions has nonnegative entries summing to *one*. (Hint: sum the entries on both sides of $\mathbf{M}\mathbf{v} = \lambda\mathbf{v}$, remembering what the columns of $\mathbf{M}$ sum to.)

Problem 20 (??pts)

Let $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ be an *orthonormal* basis of $\mathbb{R}^3$, and let

$$\mathbf{x} = 2\mathbf{q}_1 - \mathbf{q}_2 + 3\mathbf{q}_3.$$

- (A) What is $\langle \mathbf{x}, \mathbf{q}_2 \rangle$? No matrix or coordinate computation is needed.
- (B) What is $\|\mathbf{x}\|^2$?
- (C) For a *general* basis, finding the coordinates of a vector requires solving a linear system. Why is no solving needed here?
- (D) Stack the basis into the matrix $\mathbf{Q} = [\mathbf{q}_1 \ \mathbf{q}_2 \ \mathbf{q}_3]$. What is $\mathbf{Q}^\top \mathbf{Q}$? Explain.