COS 330: Great Ideas in Theoretical Computer Science

Fall 2025

Precept 11

My finale

Practice

Problem 1: Metric TSP Approximation

Consider the Traveling Salesman Problem (TSP) on a complete graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}_{\geq 0}$ satisfying the **triangle inequality**: $w(u, v) \leq w(u, x) + w(x, v)$ for all vertices $u, v, x \in V$.

The goal is to find a Hamiltonian cycle (a tour visiting every vertex exactly once) of minimum total weight. Let OPT denote this minimum weight.

An **Eulerian cycle** in a (multi)graph is a cycle that uses every edge exactly once. A connected (multi)graph has an Eulerian cycle if and only if every vertex has even degree.

Consider the following **Double-Tree Algorithm**:

1. Compute a minimum spanning tree (MST) T of G.
2. Create a multigraph G' by duplicating every edge in T (so each edge of T appears twice in G').
3. Find an Eulerian cycle C in G'. This exists because every vertex in G' has even degree.
4. Convert C to a Hamiltonian cycle H by walking along C and "short-cutting": whenever the walk reaches a vertex that has already been visited, skip it and go directly to the next unvisited vertex.

(a) Let $w(T)$ denote the total weight of edges in the MST T. Prove that $w(T) \leq \text{OPT}$.

(b) Prove that the Hamiltonian cycle H produced by the algorithm satisfies $w(H) \leq 2 \cdot \mathrm{OPT}$.