Precept Outline
- Linear-time sorting
  - Radix sort
- Probability and Concentration

Relevant Book Sections
- Book chapters: 2.1, 2.2 and 2.5

A. Advanced Precept Problems

Part 1: Sorting in Linear Time

Design a $\Theta(n)$-time algorithm for sorting arrays of length n whose elements are Java integers in $[0, n)$.

Generalize the algorithm above to *stably* sort objects with integer *keys* between 0 and $n - 1$.

Part 2: Radix Sort

Design a $\Theta(n)$-time algorithm for sorting arrays of length n whose elements are Java integers in $[0, n^2)$.
Hint: consider running the algorithm above twice with different keys.

Design a $\Theta(nB)$-time algorithm for sorting arrays of length n whose elements are integers in the range $[0, 2^B)$. Assume that obtaining the i-th bit of the binary representation of any such integer takes constant time for all $0 \leq i < B$.

B. Probability and Concentration

Let X be a discrete random variable with non-negative integer values (i.e., X is described by the probabilities $\mathbb{P}[X = k]$ for all $k \in \mathbb{N}$, which lie in $[0, 1]$ and sum to exactly 1).

We define the *expected value* of X as

$$\mathbb{E}[X] \coloneqq \sum_{i=0}^{\infty} \mathbb{P}[X = i] \cdot i.$$

Prove the following key properties of expected values:

- Linearity of expectation: for any pair X, Y of random variables, $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.
- For any pair X, Y of *independent* random variables (such that $\mathbb{P}[(X, Y) = (i, j)] = \mathbb{P}[X = i] \cdot \mathbb{P}[Y = j]$ for all i, j), the expectation of the product is the product of expectations: $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

Prove *Markov's inequality*: for every $t > 0$,

$$\mathbb{P}[X \geq t] \leq \frac{\mathbb{E}[X]}{t}.$$

The *variance* of X is defined as

$$\mathrm{Var}[X] \coloneqq \mathbb{E}[(X - \mathbb{E}[X])^2].$$

Prove *Chebyshev's inequality*: for every $t > 0$,

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \geq t\right] \leq \frac{\mathrm{Var}[X]}{t^2}.$$

In other words, prove that X concentrates around the mean with a quadratic tail bound. *Hint: apply Markov's inequality to a well-chosen random variable.*