https://algs4.cs.princeton.edu

4. Graphs and Digraphs II

- *breadth-first search (in directed graphs)*
- *breadth-first search (in undirected graphs)*
- *topological sort*
- *challenges*

Last updated on 3/27/25 7:37AM

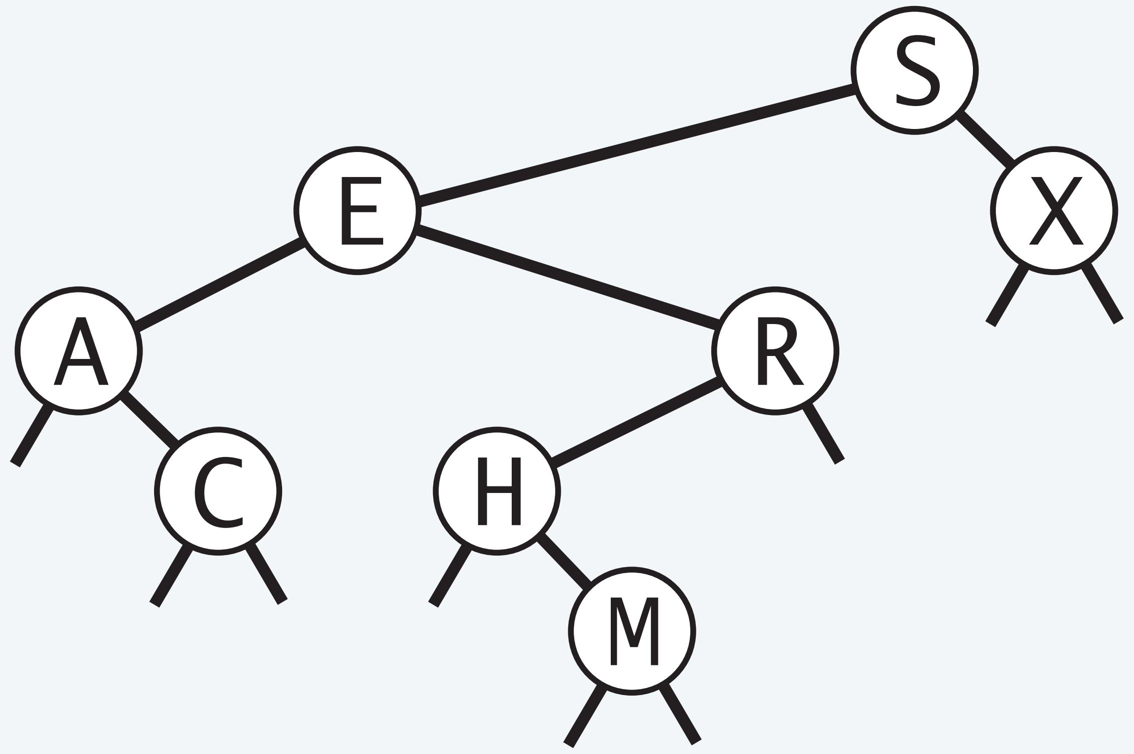
Graph search overview

Tree traversal. Many ways to systematically explore nodes in a binary tree.

- Inorder: A C E H M R S X
- Preorder: S E A C R H M X
- Postorder: C A M H R E X S
- Level-order: S E X A R C H M

stack/recursion (Inorder, Preorder, Postorder)

queue (Level-order)

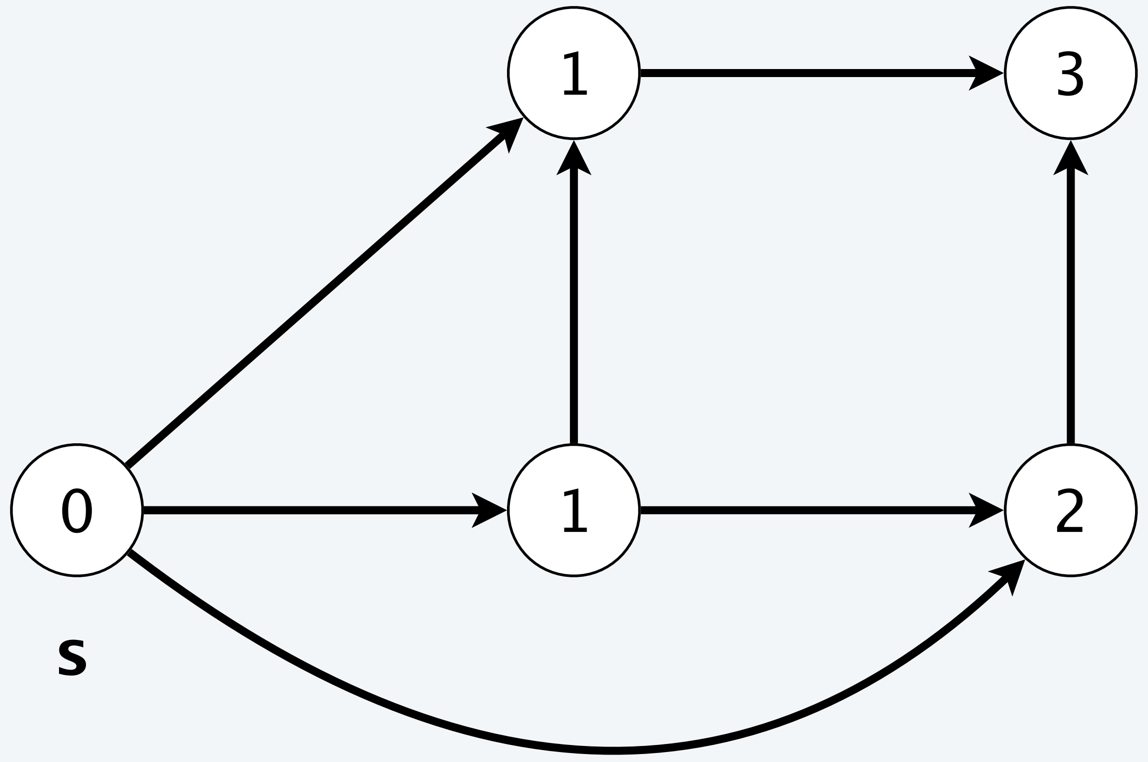


Graph traversal. Many ways to systematically explore vertices in a graph or digraph.

- DFS preorder: vertices in order of calls to `dfs(G, v)`.
- DFS postorder: vertices in order of returns from `dfs(G, v)`.
- BFS order: vertices in increasing order of distance from `s`.

stack/recursion (DFS preorder, DFS postorder)

queue (BFS order)



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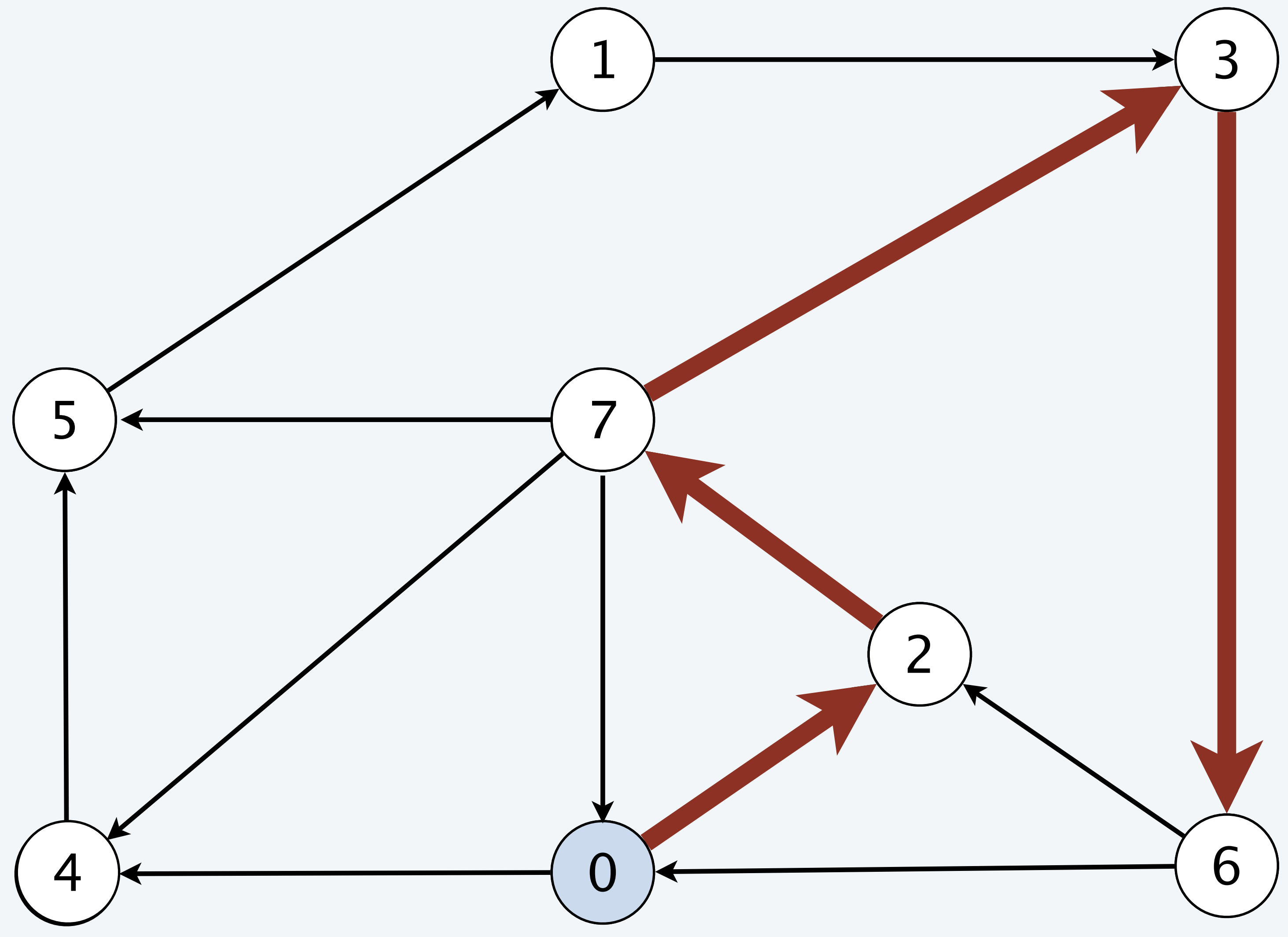
4. Graphs and Digraphs II

- breadth-first search (in directed graphs)
- breadth-first search (in undirected graphs)
- topological sort
- challenges

Shortest paths in a digraph

Problem. Find directed path from s to each other vertex that uses the fewest edges.



directed paths from 0 to 6

$0 \to 2 \to 7 \to 4 \to 5 \to 1 \to 3 \to 6$

$0 \to 4 \to 5 \to 1 \to 3 \to 6$

$0 \to 2 \to 7 \to 3 \to 6$

$0 \to 2 \to 7 \to 0 \to 2 \to 7 \to 3 \to 6$

shortest path must be simple (no repeated vertices)

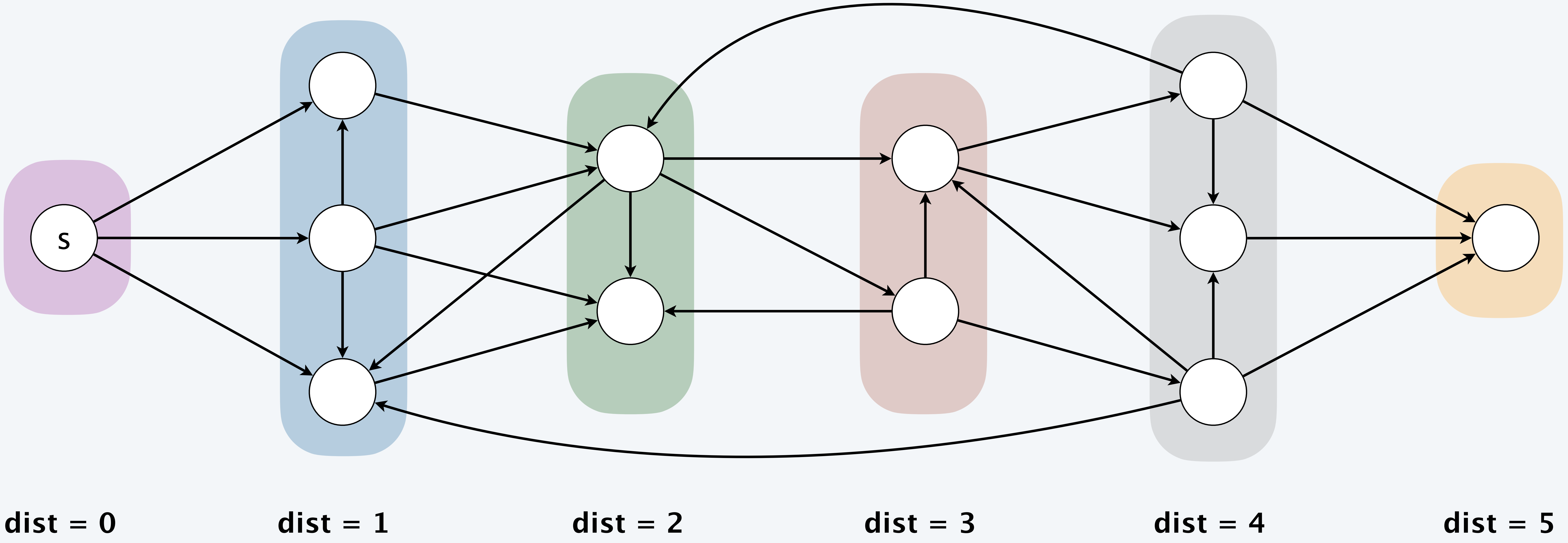
shortest path from 0 to 6 (length = 4)

$0 \to 2 \to 7 \to 3 \to 6$

Shortest paths in a digraph

Problem. Find directed path from s to each other vertex that uses the fewest edges.

Key idea. Visit vertices in increasing order of distance from s.



Q. How to implement?

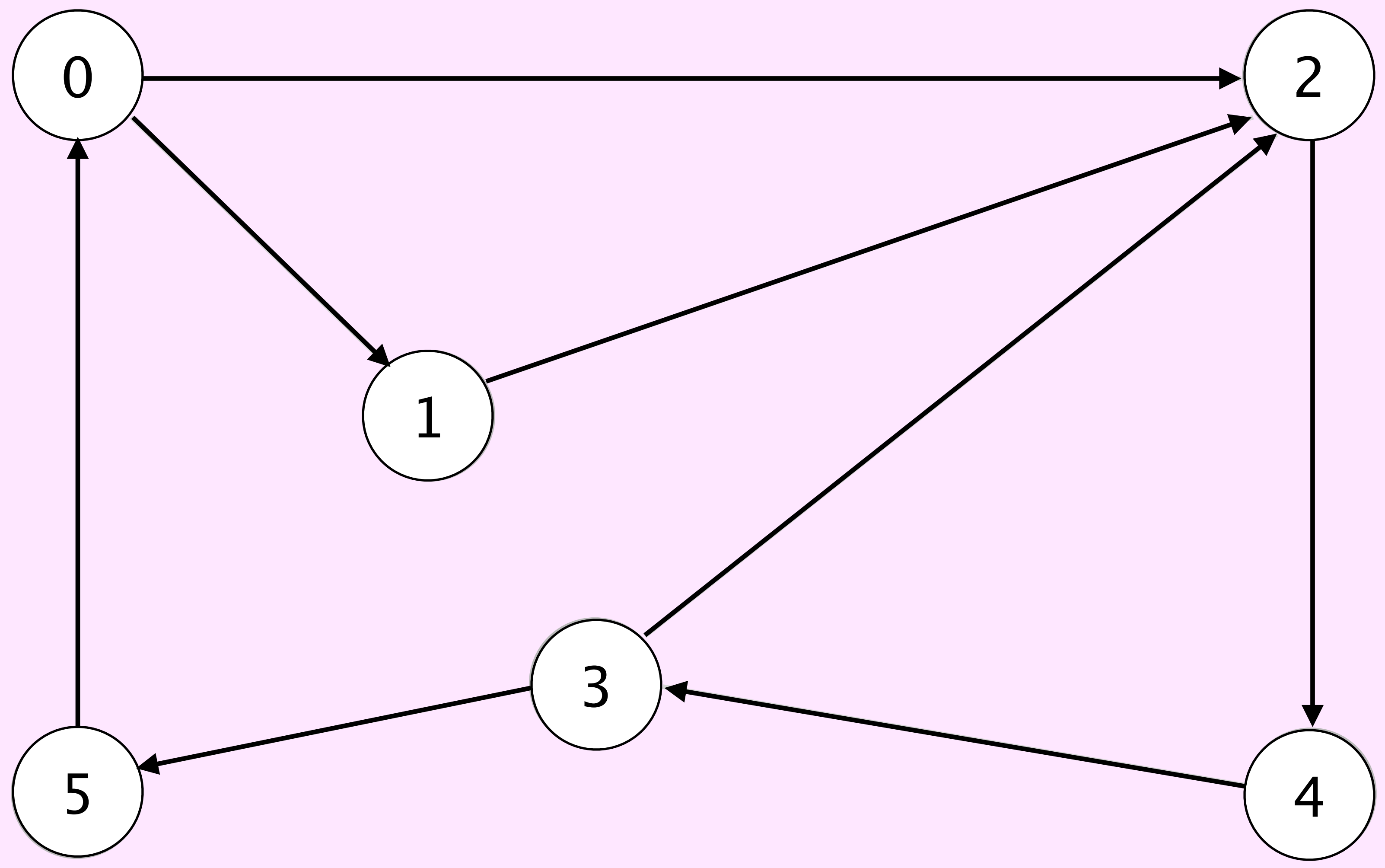
A. Store vertices to visit in a queue.

Breadth-first search (in a digraph) demo

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent from v and mark them.

visit vertex v



graph G

tinyDG2.txt

V
E

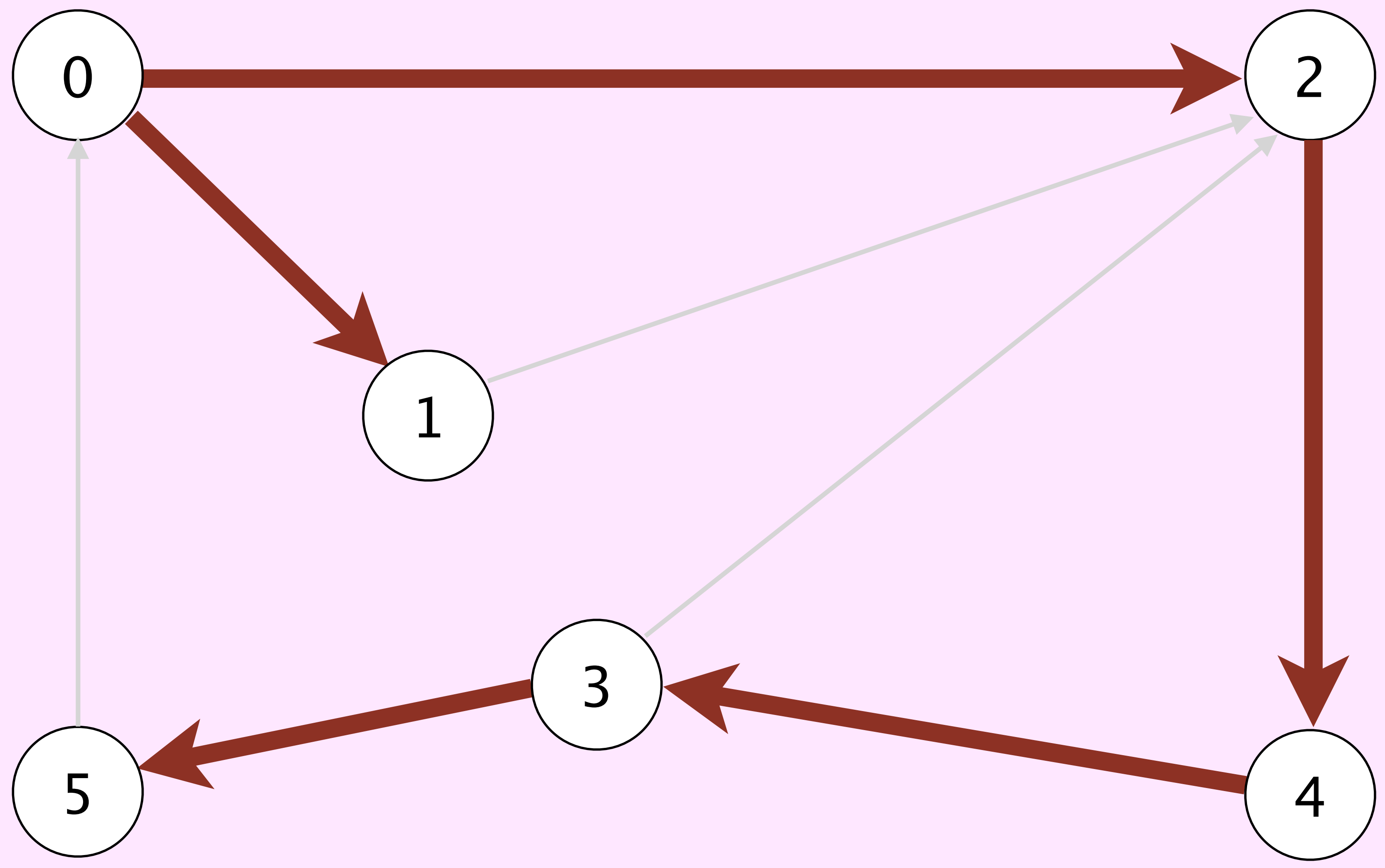
```
6
8
5 0
2 4
3 2
1 2
0 1
4 3
3 5
0 2
```

Breadth-first search (in a digraph) demo

Repeat until queue is empty:

- Remove vertex *v* from queue.
- Add to queue all unmarked vertices adjacent from *v* and mark them.

visit vertex v



v	edgeTo[]	marked[]	distTo[]
0	–	T	0
1	0	T	1
2	0	T	1
3	4	T	3
4	2	T	2
5	3	T	4

vertices reachable from 0 (and shortest directed paths)

Breadth-first search

Repeat until queue is empty:

- Remove vertex v from queue.
- Add to queue all unmarked vertices adjacent from v and mark them.

visit vertex v

BFS (from source vertex s)

Add vertex s to FIFO queue and mark s.

Repeat until the queue is empty:

- **remove the least recently added vertex v**
- **for each unmarked vertex w adjacent from v:**
  add w to queue and mark w

Breadth-first search: Java implementation

```
public class BreadthFirstDirectedPaths {
   private boolean[] marked;
   private int[] edgeTo;
   private int[] distTo;
   ...

   private void bfs(Digraph G, int s) {
      Queue<Integer> queue = new Queue<>();
      queue.enqueue(s);
      marked[s] = true;
      distTo[s] = 0;

      while (!queue.isEmpty()) {
         int v = queue.dequeue();
         for (int w : G.adj(v)) {
            if (!marked[w]) {
               queue.enqueue(w);
               marked[w] = true;
               edgeTo[w] = v;
               distTo[w] = distTo[v] + 1;
            }
         }
      }
   }

}
```

initialize queue of vertices to explore

also safe to stop as soon as all vertices marked

found new vertex w via edge v→w

https://algs4.cs.princeton.edu/42digraph/BreadthFirstDirectedPaths.java.html

Breadth-first search properties

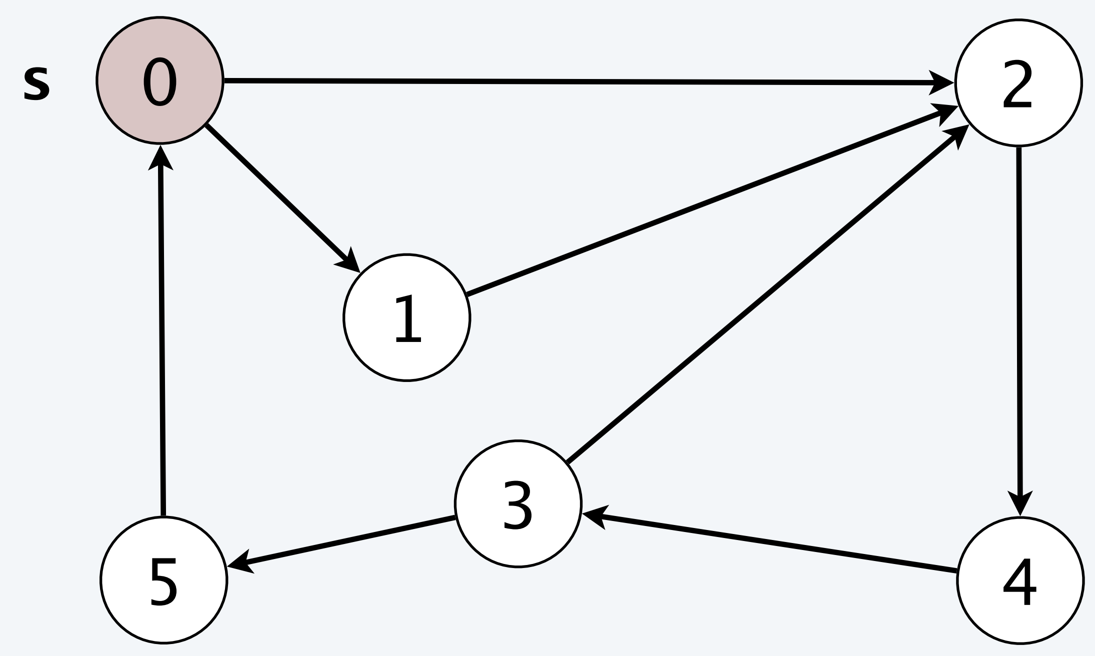
Proposition. In the worst case, BFS takes $\Theta(E + V)$ time.

Pf. Each vertex reachable from s is visited once.

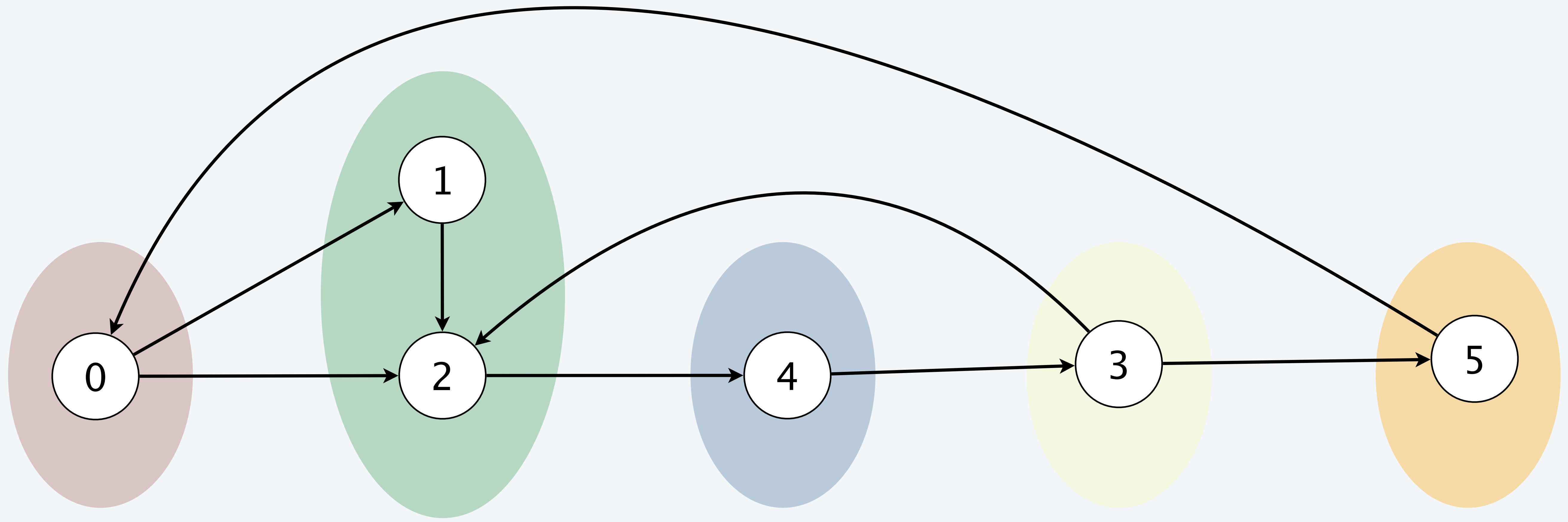
Proposition. BFS computes shortest paths from s.

Pf idea. BFS examines vertices in increasing order of distance (number of edges) from s.

invariant: queue contains some vertices of distance k from s,
followed by ≥ 0 vertices of distance k+1 (and no other vertices)



digraph G



dist = 0 dist = 1 dist = 2 dist = 3 dist = 4

Graphs and digraphs II: poll 1

What could happen if we mark a vertex when it is dequeued (instead of enqueued)?

- **A.** Doesn't find a shortest path.
- **B.** Takes exponential time.
- **C.** Both A and B.
- **D.** Neither A nor B.

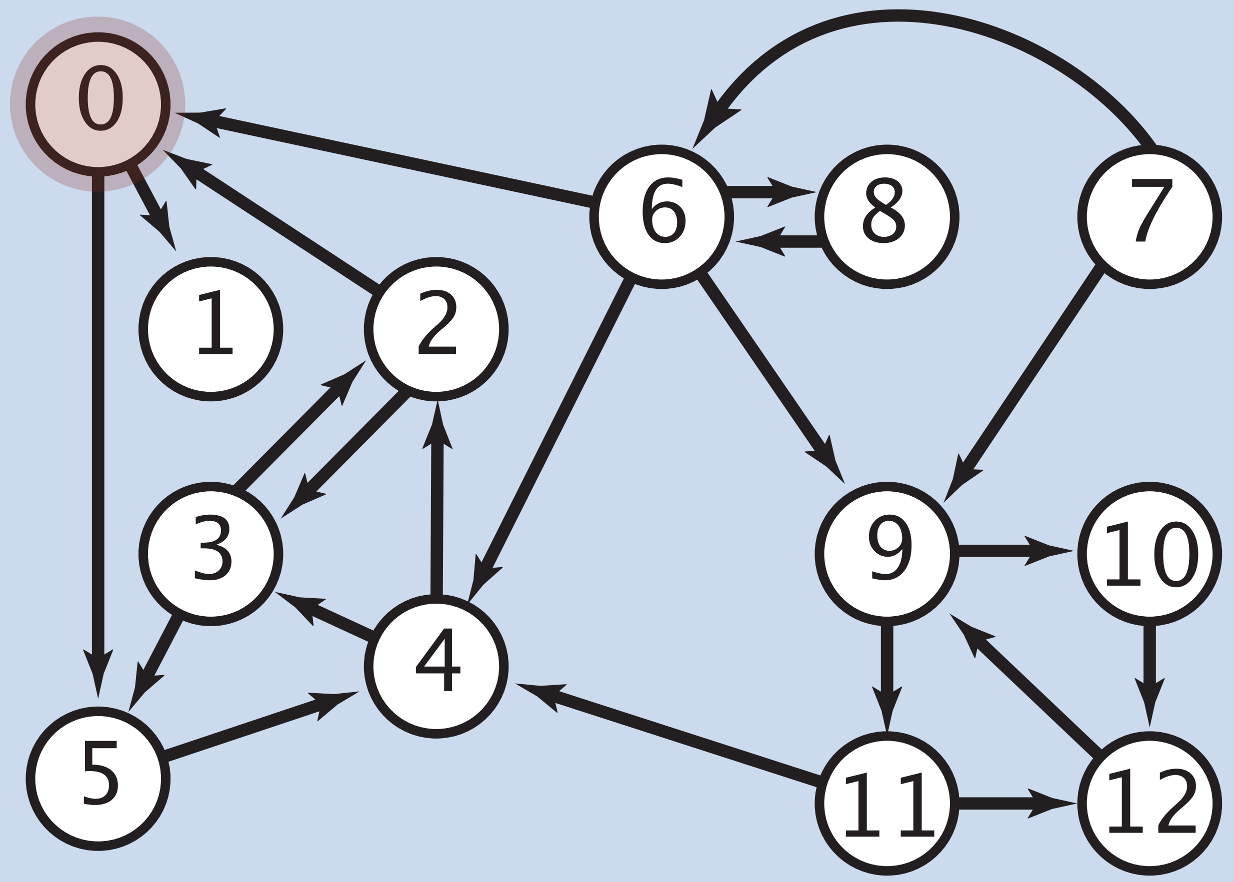
```
while (!queue.isEmpty()) {
   int v = queue.dequeue();
   marked[v] = true;
   for (int w : G.adj(v)) {
      if (!marked[w]) {
         marked[w] = true;
         queue.enqueue(w);
         edgeTo[w] = v;
         distTo[w] = distTo[v] + 1;
      }
   }
}
```

Single-target shortest paths

Given a digraph and a target vertex t, find shortest path from every vertex to t.

Ex. $t = 0$

- Shortest path from 7 is 7→6→0.
- Shortest path from 5 is 5→4→2→0.
- Shortest path from 12 is 12→9→11→4→2→0.
- …



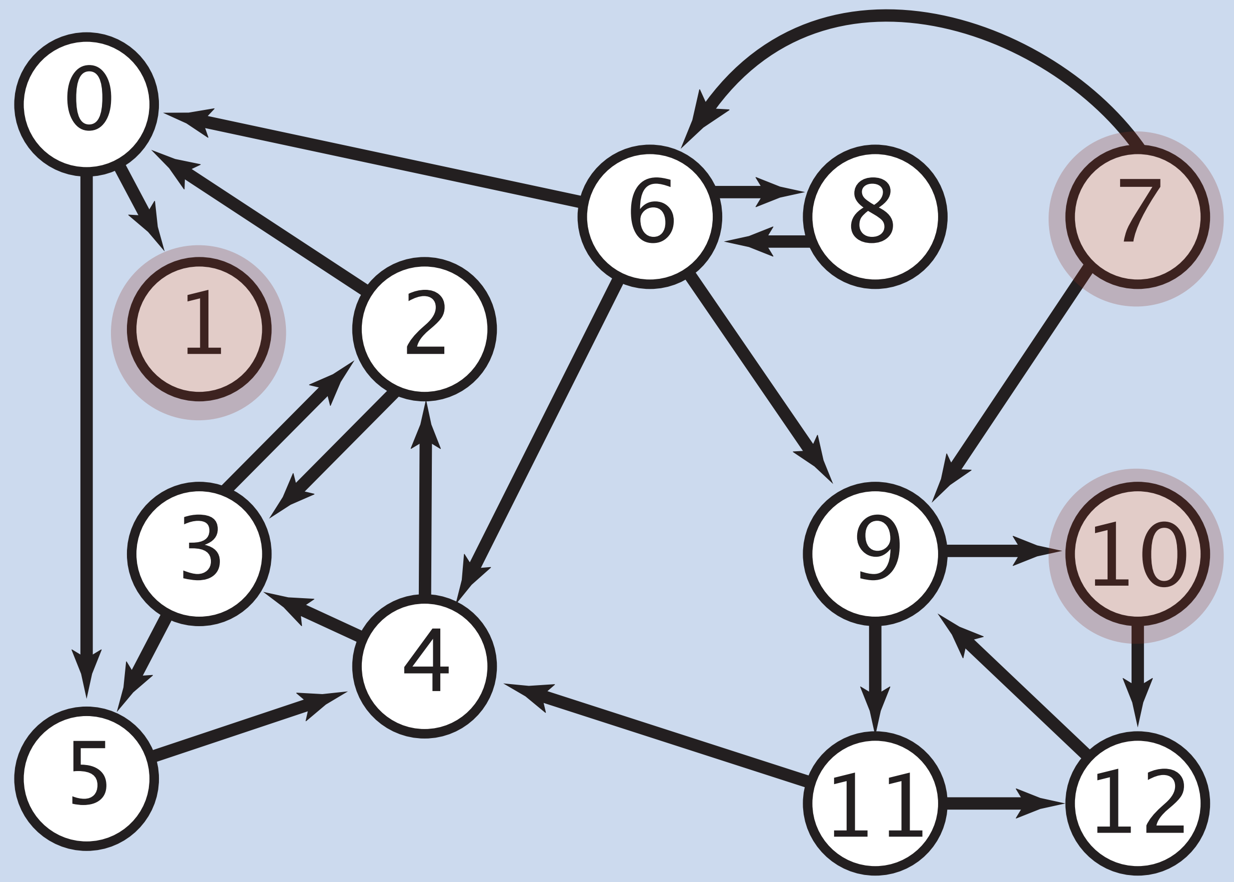
Q. How to implement single-target shortest paths algorithm?

Multiple-source shortest paths

Given a digraph and a set of source vertices, find shortest path from any vertex in the set to every other vertex.

Ex. $S = \{ 1, 7, 10 \}$.

- Shortest path to 4 is 7→6→4.
- Shortest path to 5 is 7→6→0→5.
- Shortest path to 12 is 10→12.
- …



needed for WordNet assignment

Q. How to implement multi-source shortest paths algorithm?

Graphs and digraphs II: poll 2

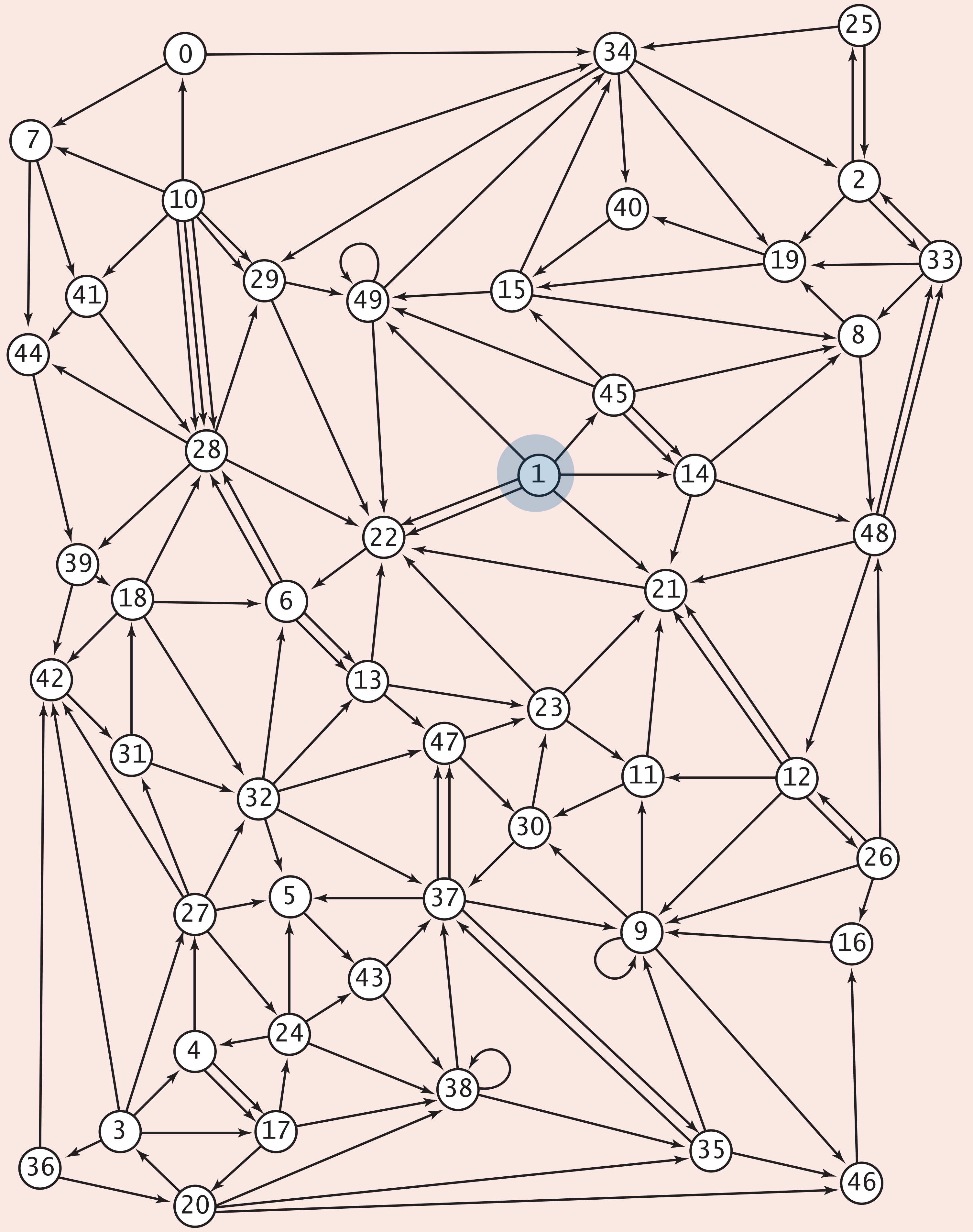
Suppose that you want to design a web crawler. Which core algorithm should you use?

A. Depth-first search.

B. Breadth-first search.

C. Either A or B.

D. Neither A nor B.



Web crawler output

BFS crawl

```
https://www.princeton.edu
https://www.w3.org
https://ogp.me
https://giving.princeton.edu
https://www.princetonartmuseum.org
https://www.goprincetontigers.com
https://library.princeton.edu
https://helpdesk.princeton.edu
https://tigernet.princeton.edu
https://alumni.princeton.edu
https://gradschool.princeton.edu
https://vimeo.com
https://princetonusg.com
https://artmuseum.princeton.edu
https://jobs.princeton.edu
https://odoc.princeton.edu
https://blogs.princeton.edu
https://www.facebook.com
https://twitter.com
https://www.youtube.com
https://deimos.apple.com
https://qeprize.org
https://en.wikipedia.org
...
```

DFS crawl

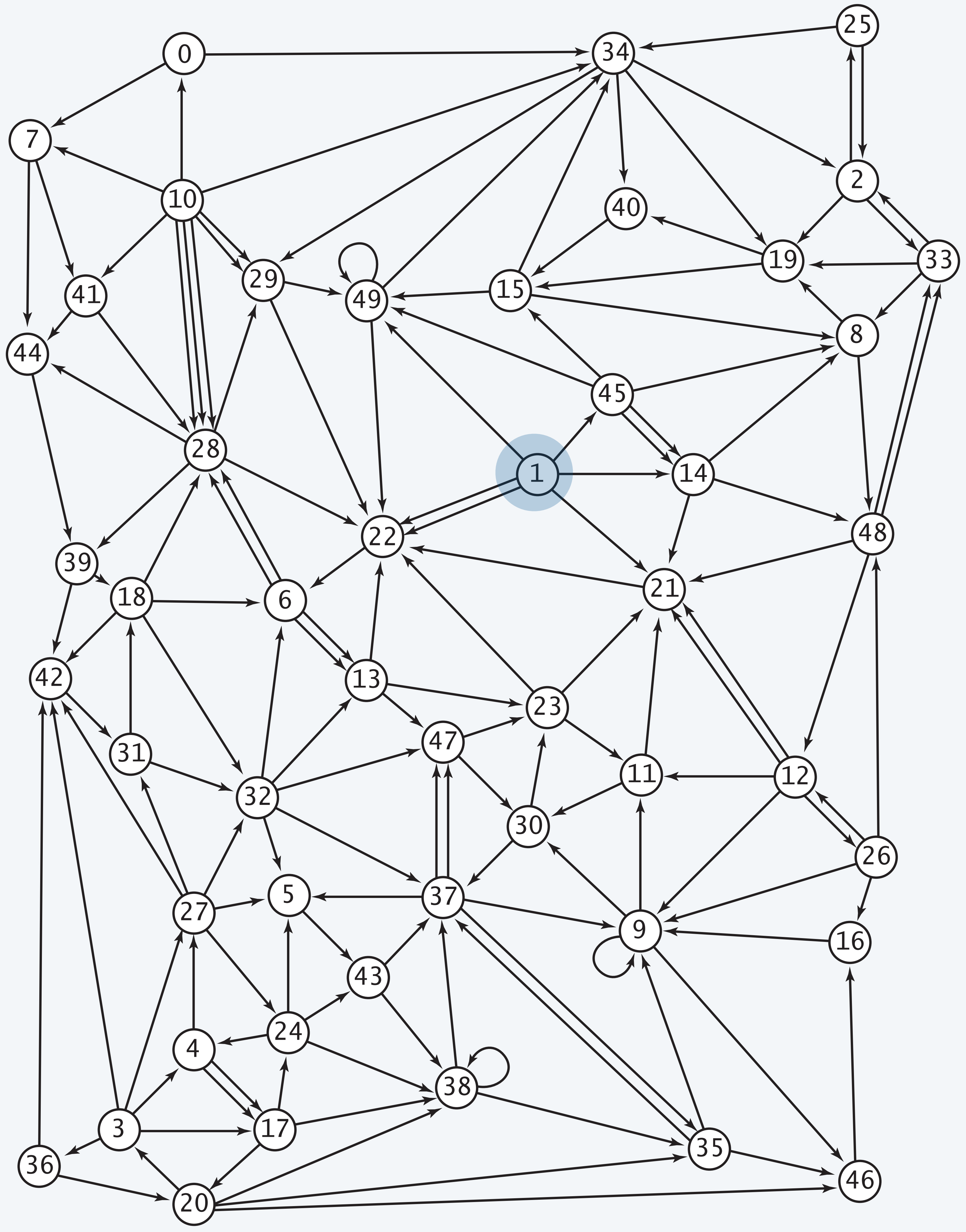
```
https://www.princeton.edu
https://deimos.apple.com
https://www.youtube.com
https://www.google.com
https://news.google.com
https://csi.gstatic.com
https://googlenewsblog.blogspot.com
https://labs.google.com
https://groups.google.com
https://img1.blogblog.com
https://feeds.feedburner.com
https://buttons.googlesyndication.com
https://fusion.google.com
https://insidesearch.blogspot.com
https://agoogleaday.com
https://static.googleusercontent.com
https://searchresearch1.blogspot.com
https://feedburner.google.com
https://www.dot.ca.gov
https://www.TahoeRoads.com
https://www.LakeTahoeTransit.com
https://www.laketahoe.com
https://ethel.tahoeguide.com
...
```

Application: web crawler

Goal. Crawl web, starting from some root web page, say `https://www.princeton.edu`.

Solution. [BFS with implicit digraph]

- Choose root web page as source s.
- Maintain a queue of websites to explore.
- Maintain a set of marked websites.
- Dequeue the next website and enqueue any unmarked websites to which it links.



Caveat. Industrial-strength web crawlers use same core idea, but more sophisticated techniques.

Bare-bones web crawler: Java implementation

```
Queue<String> queue = new Queue<>();
SET<String> marked = new SET<>();

String root = "https://www.princeton.edu";
queue.enqueue(root);
marked.add(root);

while (!queue.isEmpty()) {
    String v = queue.dequeue();
    StdOut.println(v);
    In in = new In(v);
    String input = in.readAll();

    String regexp = "https://(\\w+\\.)+(\\w+)";
    Pattern pattern = Pattern.compile(regexp);
    Matcher matcher = pattern.matcher(input);

    while (matcher.find()) {
      String w = matcher.group();

      if (!marked.contains(w)) {
          marked.add(w);
          queue.enqueue(w);
      }

    }
}
```

queue of websites to crawl

set of marked websites

start crawling from root website

read in raw HTML from next website in queue

use regular expression to find all URLs in website of form `https://xxx.yyy.zzz` *[crude pattern misses relative URLs]*

if unmarked, mark and enqueue

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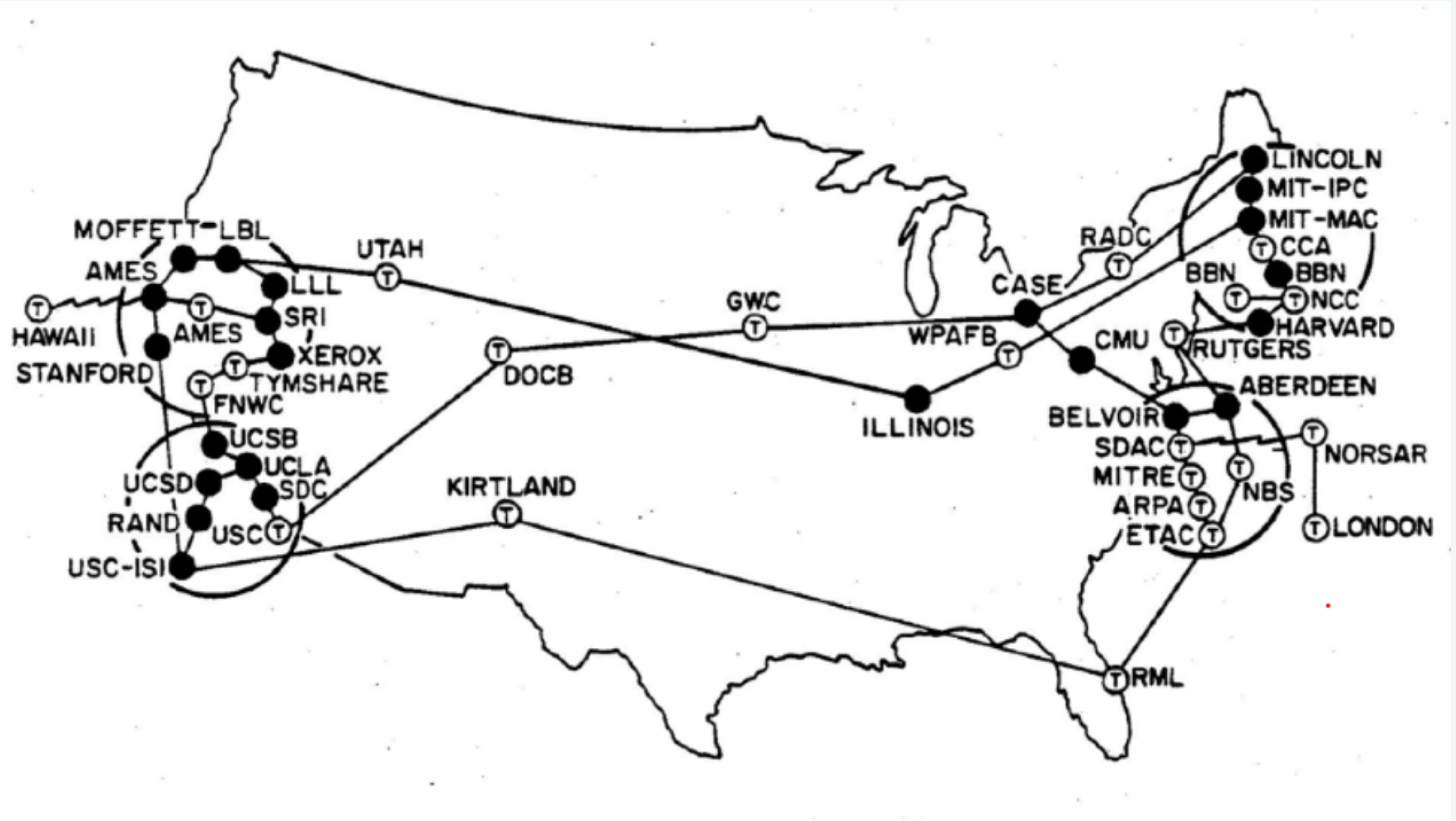
Application: routing in a communication network

Fewest number of hops in a communication network.



ARPANET 1970s

Breadth-first search in undirected graphs

Problem. Find path between s and each other vertex that uses fewest edges.

Solution. Use BFS. ← *but now, for each undirected edge v–w: v is adjacent to w, and w is adjacent to v*

BFS (from source vertex s)

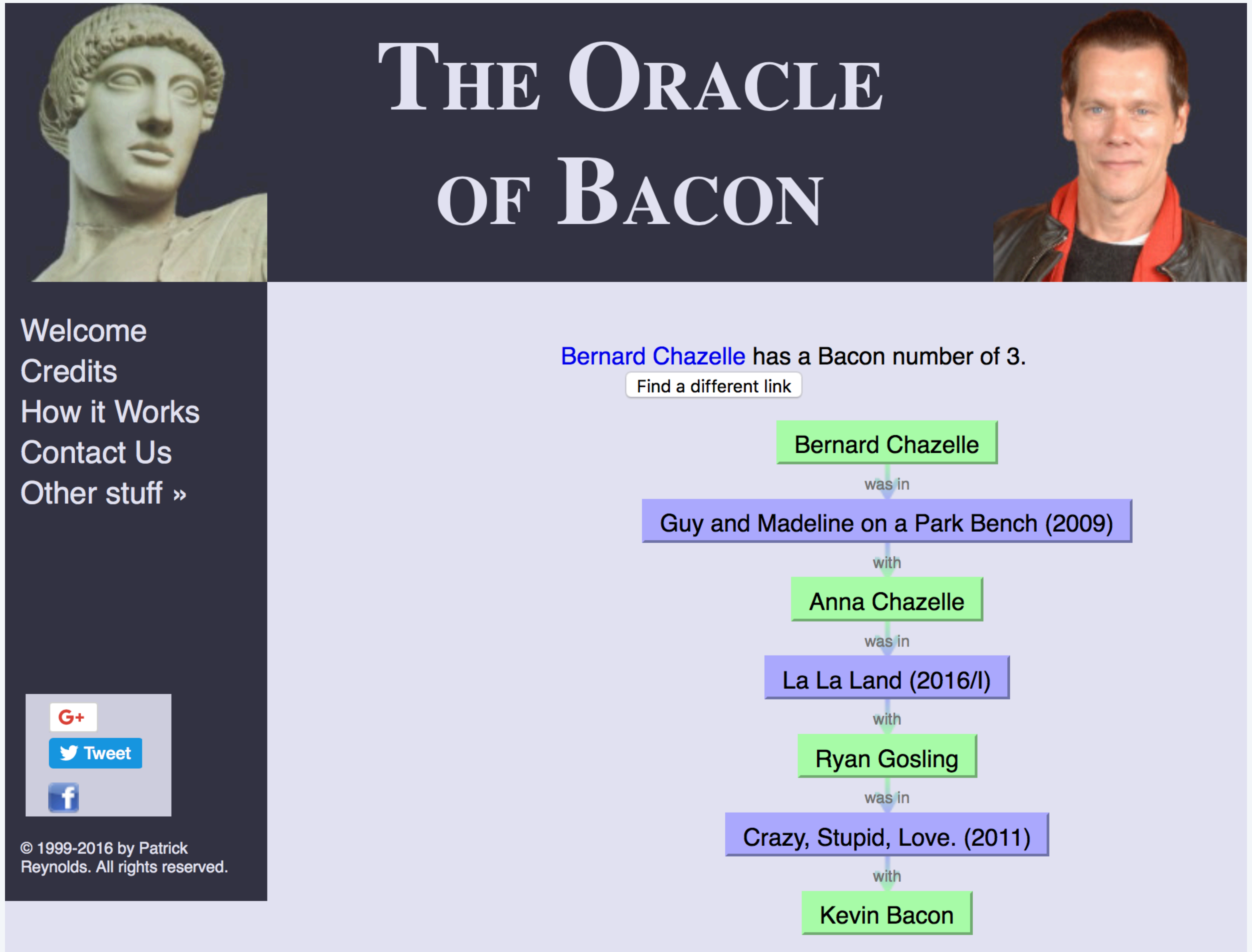
Add vertex s to FIFO queue and mark s.

Repeat until the queue is empty:

- **remove the least recently added vertex v**
- **for each unmarked vertex w adjacent to v:**
  add w to queue and mark w

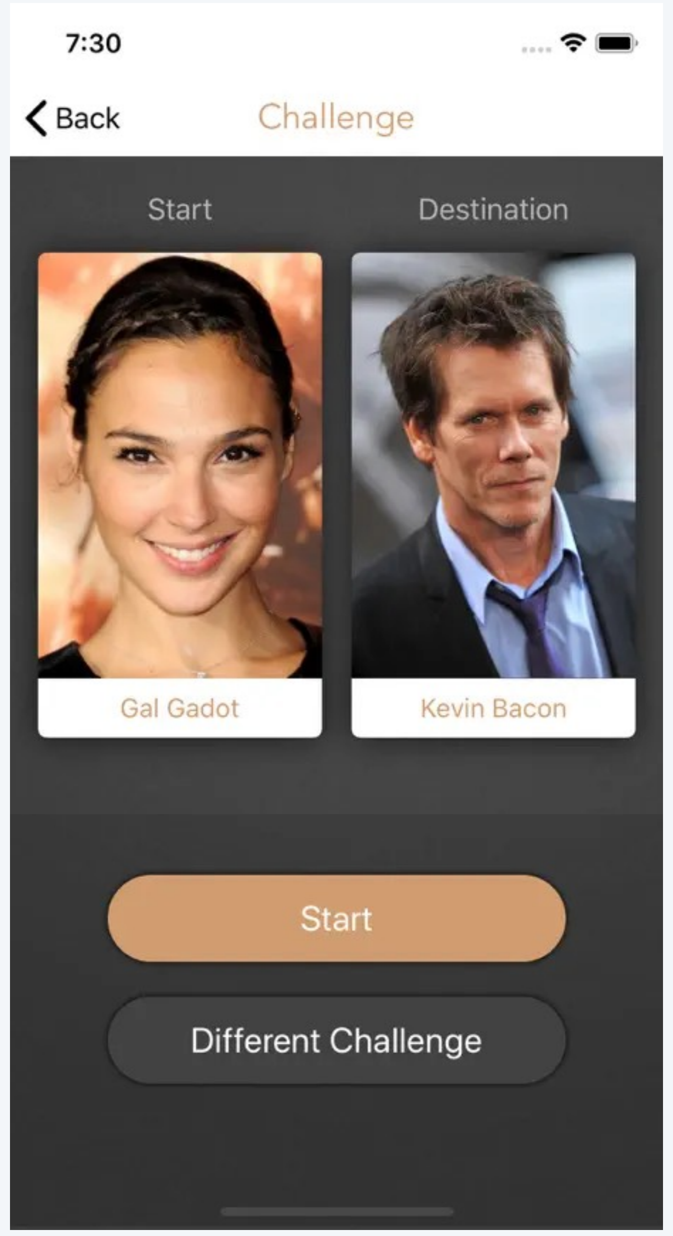
Proposition. BFS finds shortest paths between s and every other vertex in $\Theta(E + V)$ time.

Application: Kevin Bacon numbers



https://oracleofbacon.org

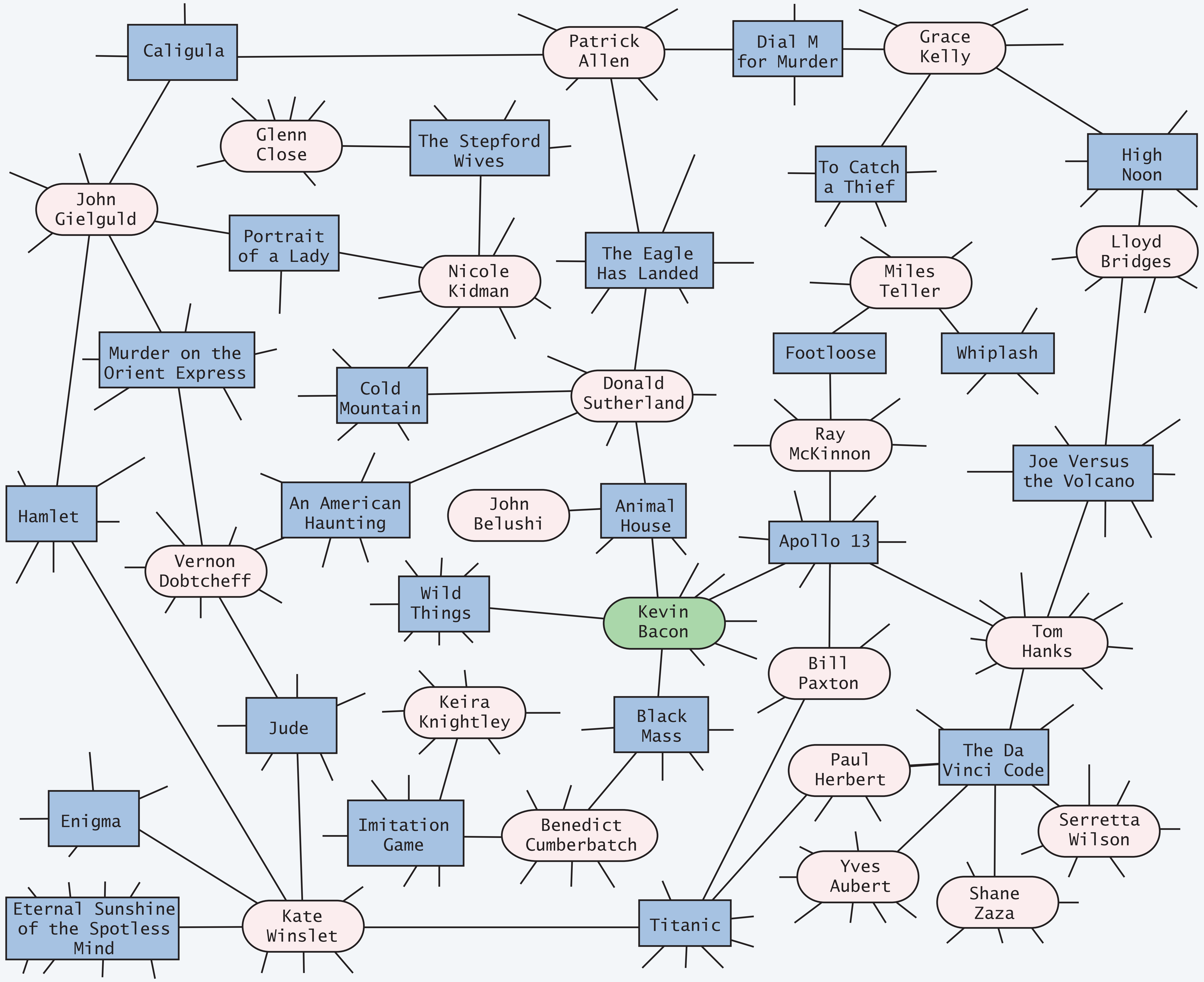
Endless Games board game



SixDegrees of Hollywood

Kevin Bacon graph

- Include one vertex for each performer and one for each movie.
- Connect a movie to all performers that appear in that movie.
- Compute shortest paths between s = Kevin Bacon and every other performer.



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4. Graphs and Digraphs II

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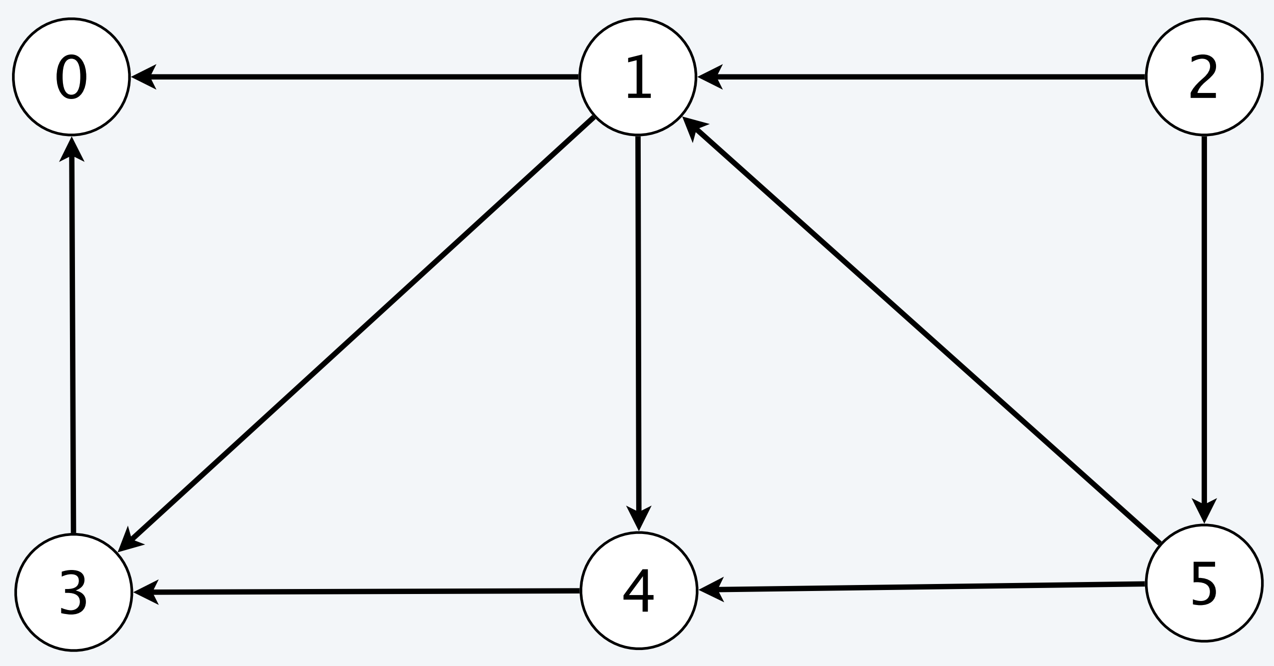
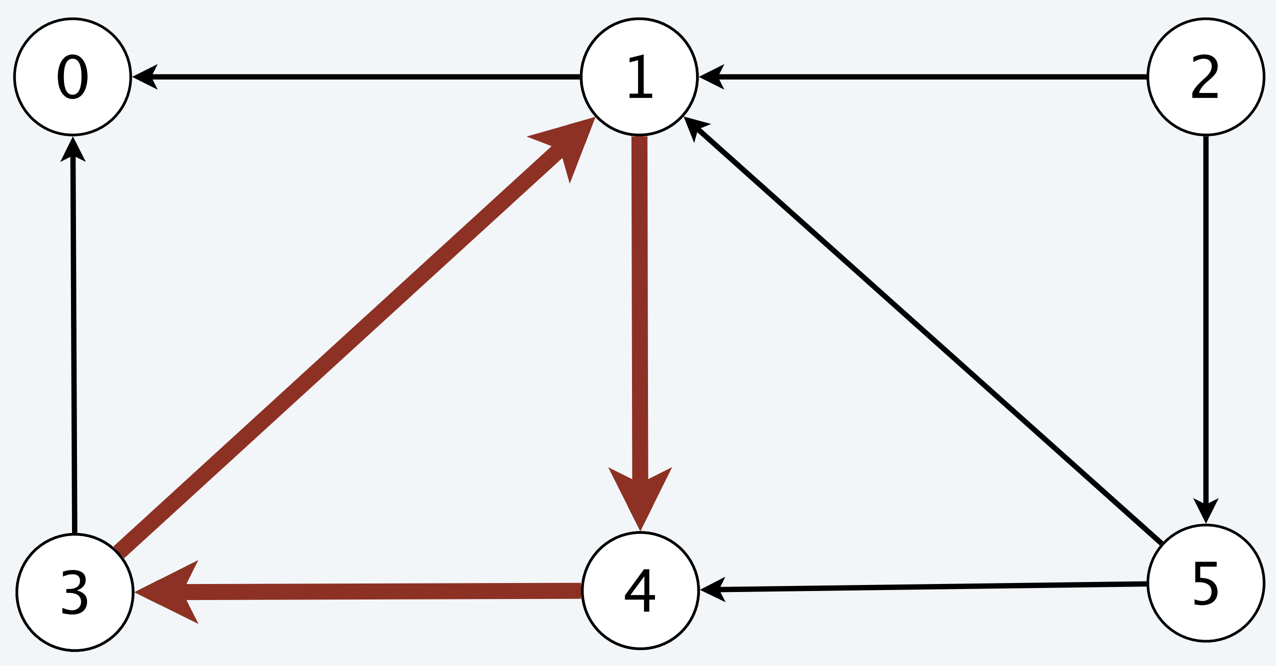
- topological sort
- challenges

Directed acyclic graphs

Directed acyclic graph (DAG). A digraph with no directed cycles.



DAG
(no directed cycles)

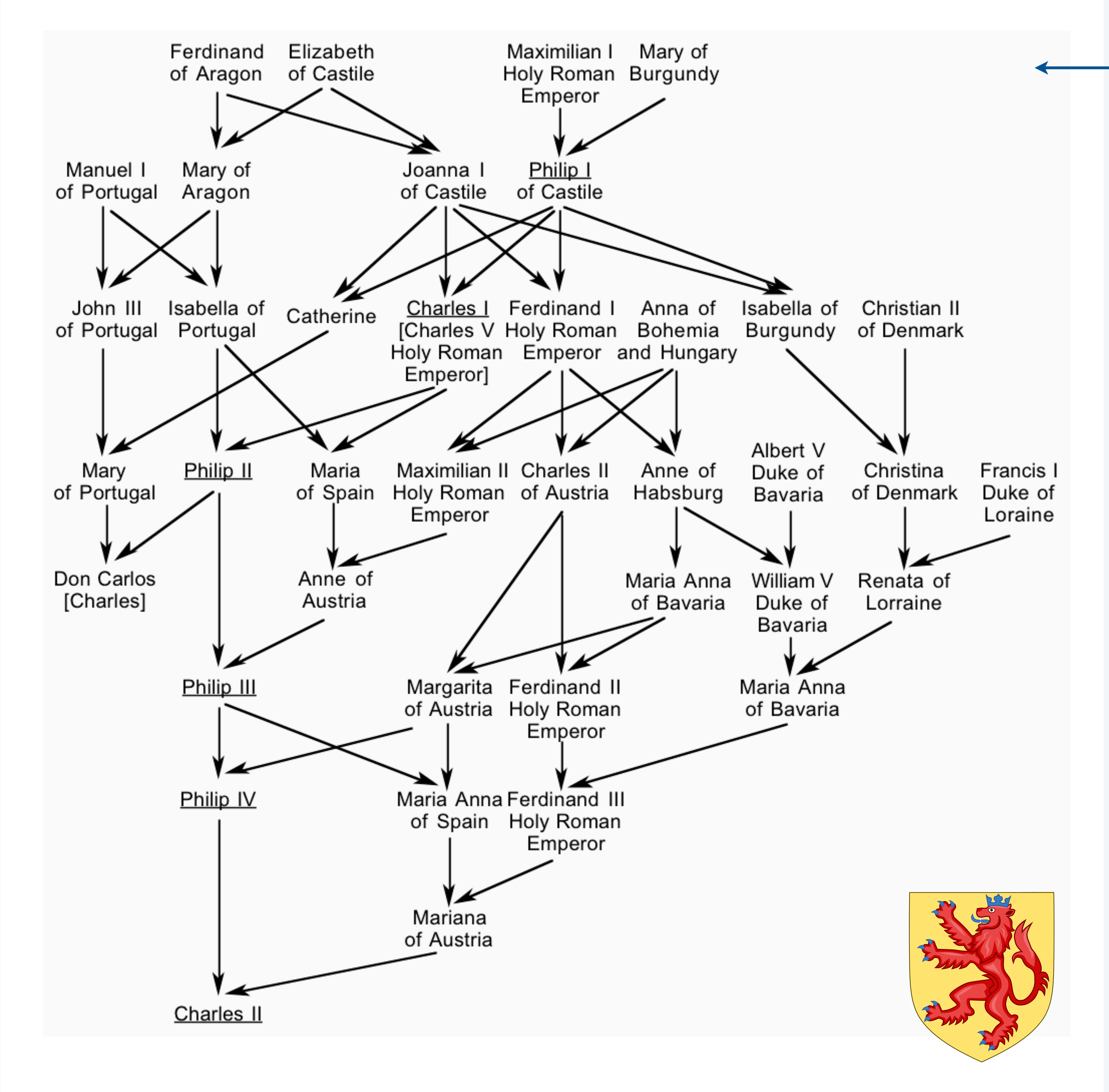


digraph
(but not a DAG)

Remark. DAGs are an important subclass of digraphs that arise in many applications.

Family ~~tree~~ DAG

Vertex = person; edge = biological child.

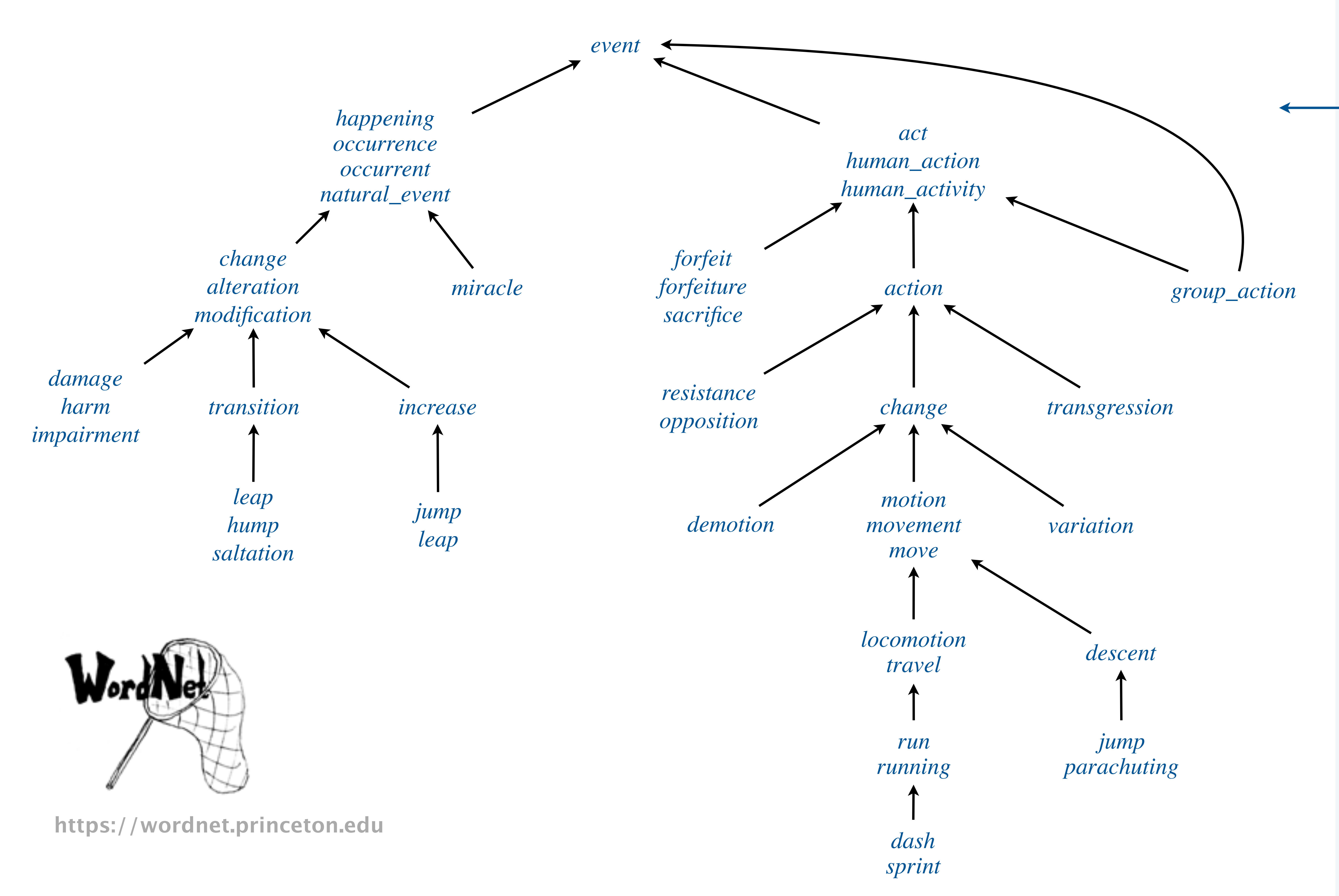


no directed cycles
(a person can't be their own ancestor)

pedigree of King Charles II of Spain

WordNet DAG

Vertex = synset; edge = hypernym relationship.

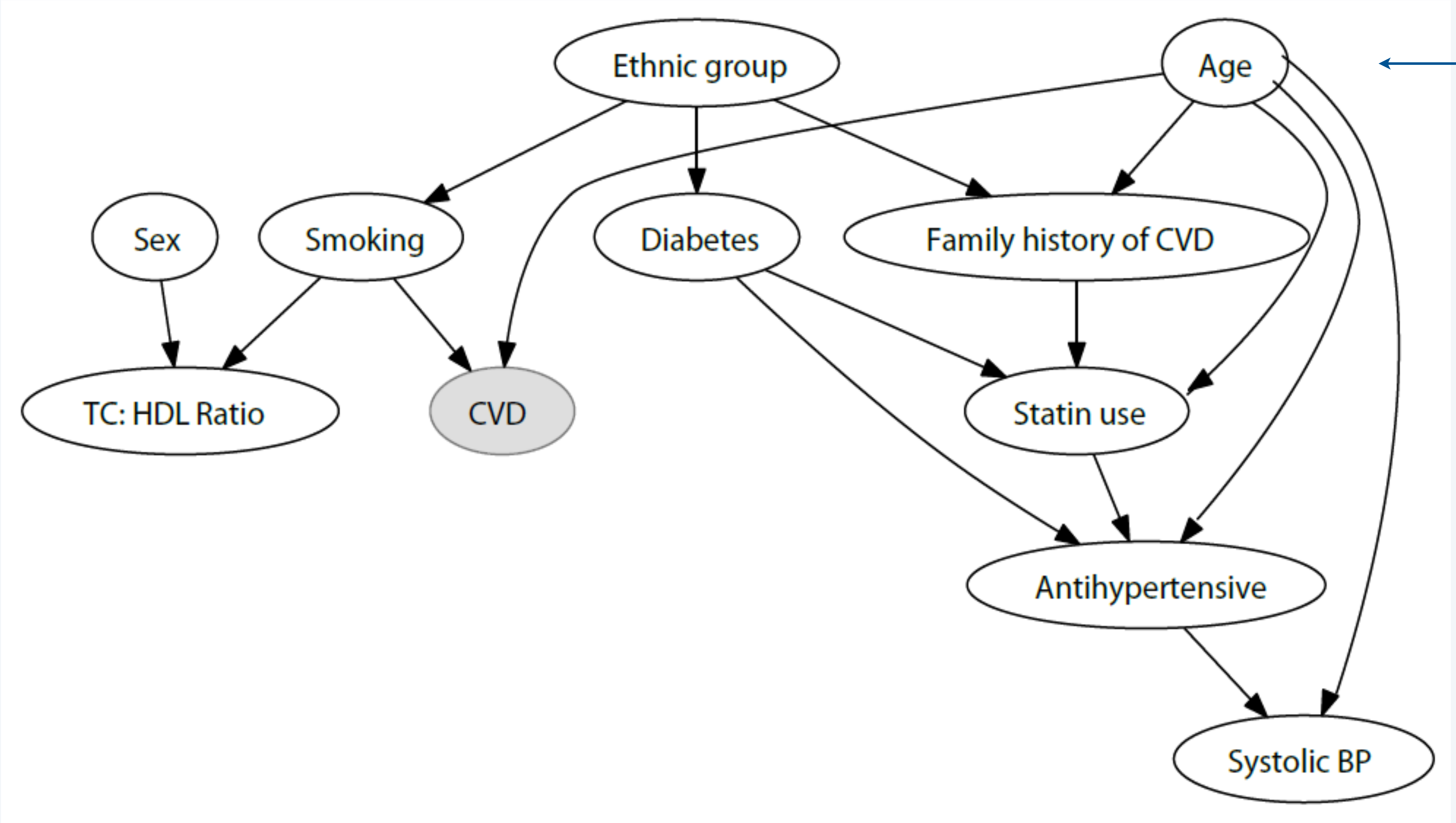


no directed cycles
(a synset can't be more general than itself)

a subgraph of the WordNet DAG

Bayesian networks

Vertex = variable; edge = conditional dependency.

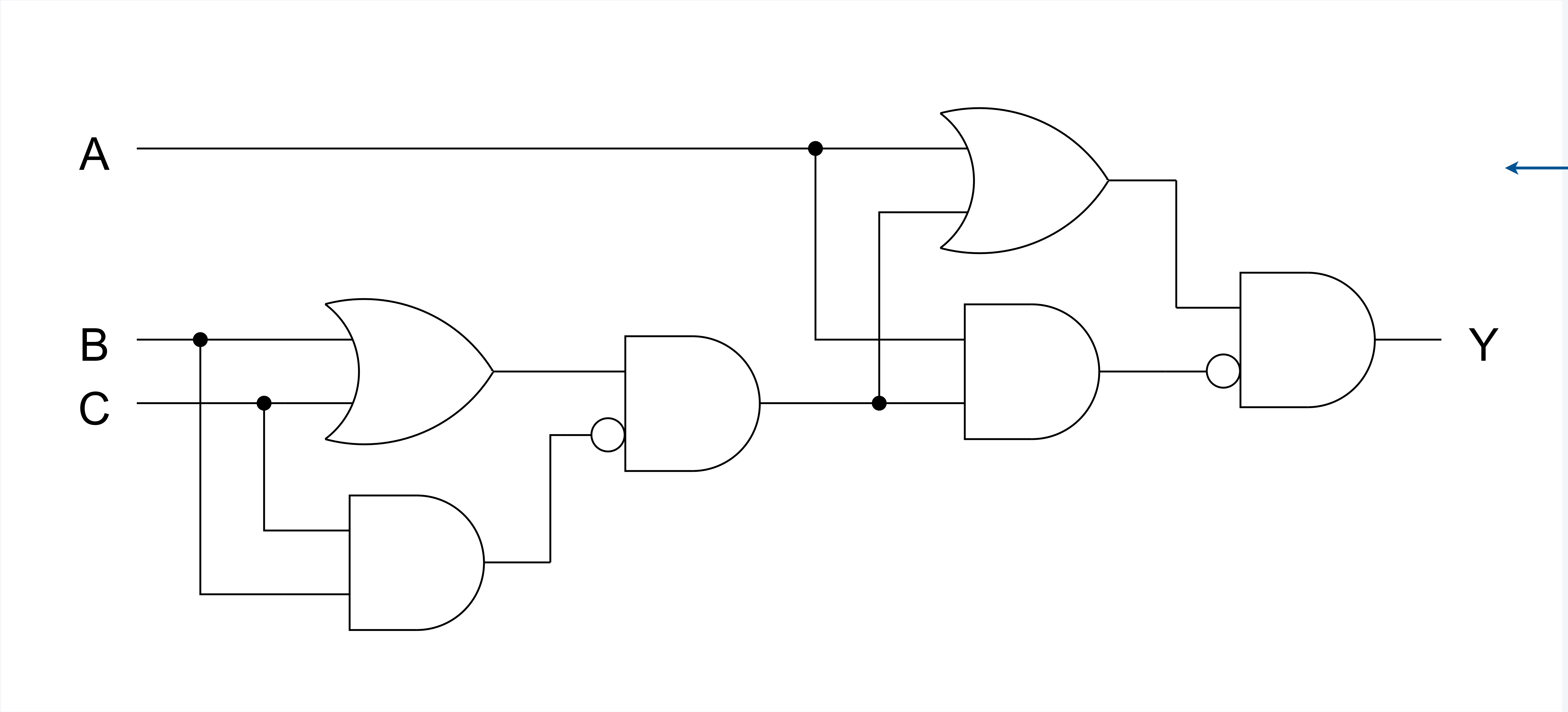


no directed cycles
(a variable can't depend upon itself)

Using DAGs for Investigating Causal Paths for Cardiovascular Disease

Combinational circuits

Digital logical circuit. Vertex = logic gate; edge = wire.



no directed cycles $\Longrightarrow$ *combinational circuit*

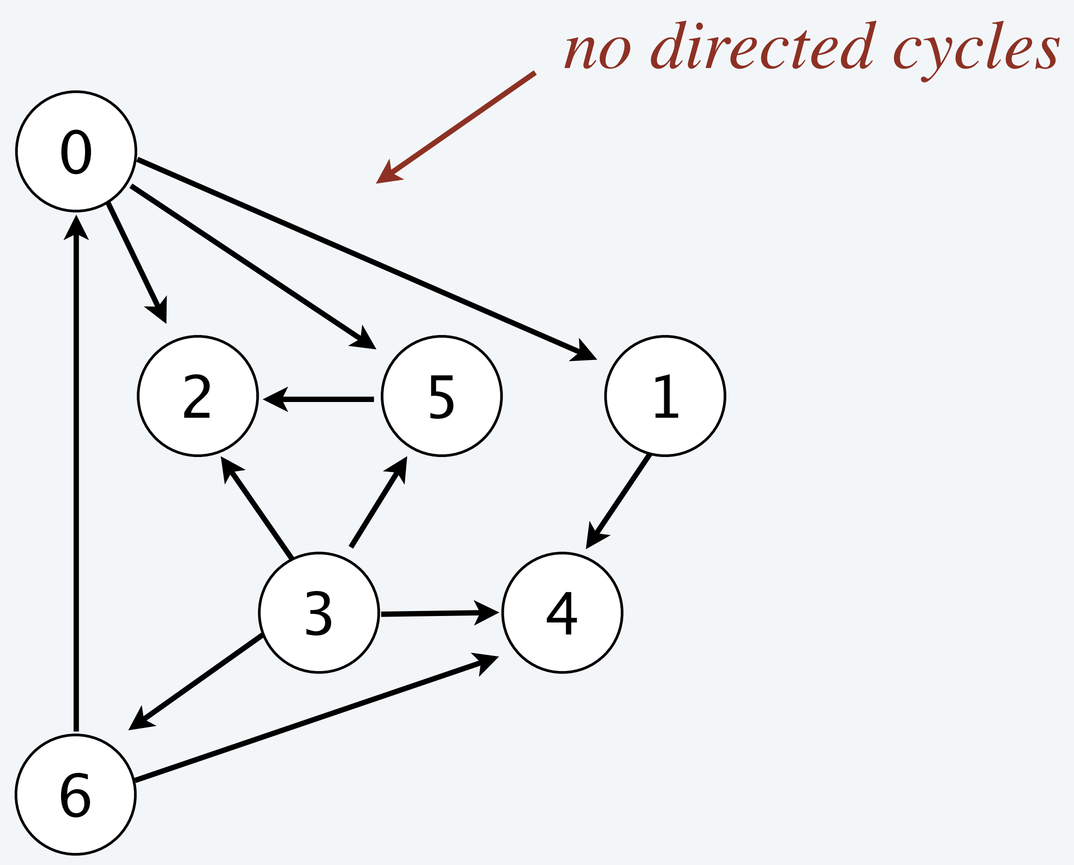
Precedence scheduling

Goal. Given a set of tasks to be completed with precedence constraints, in which order should we schedule the tasks?

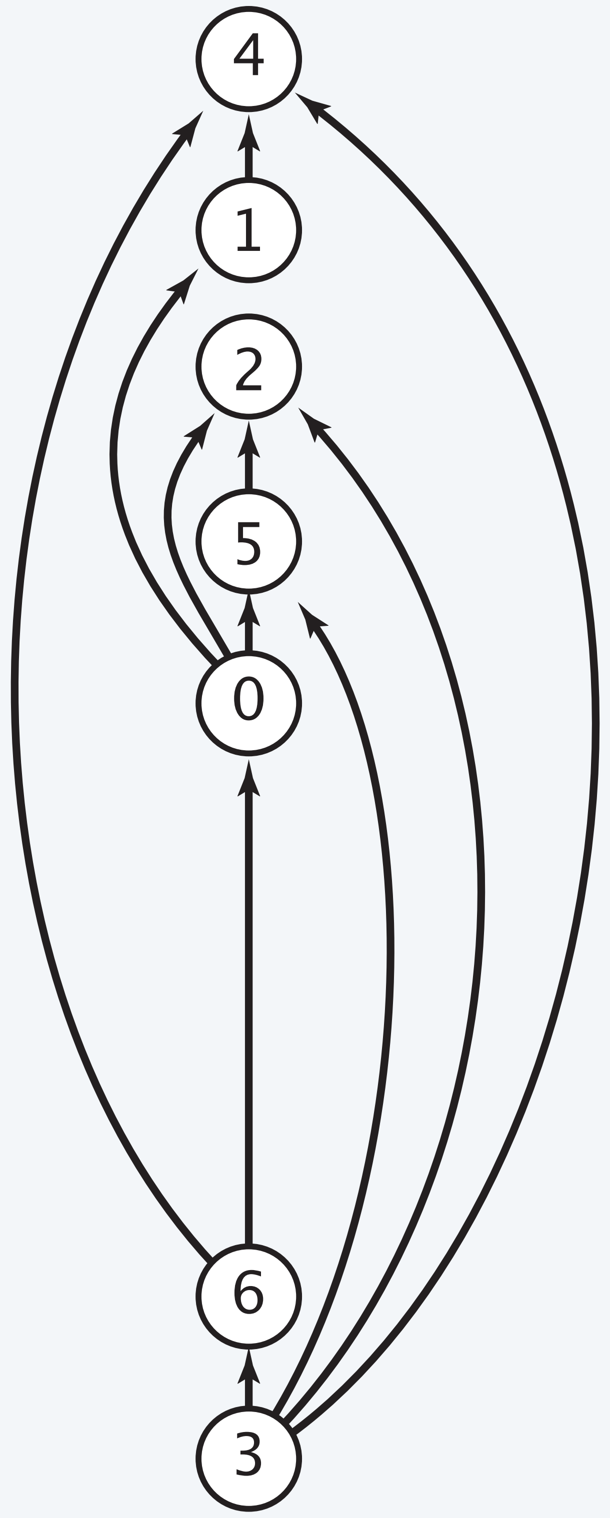
Digraph model. vertex = task; edge = precedence constraint.

0. Math for CS
1. Complexity Theory
2. Machine Learning
3. Intro to CS
4. Cryptography
5. Scientific Computing
6. Algorithms

tasks



precedence constraint graph



feasible schedule

Applications. Project management, compilers, parallel computing, ...

Topological sort

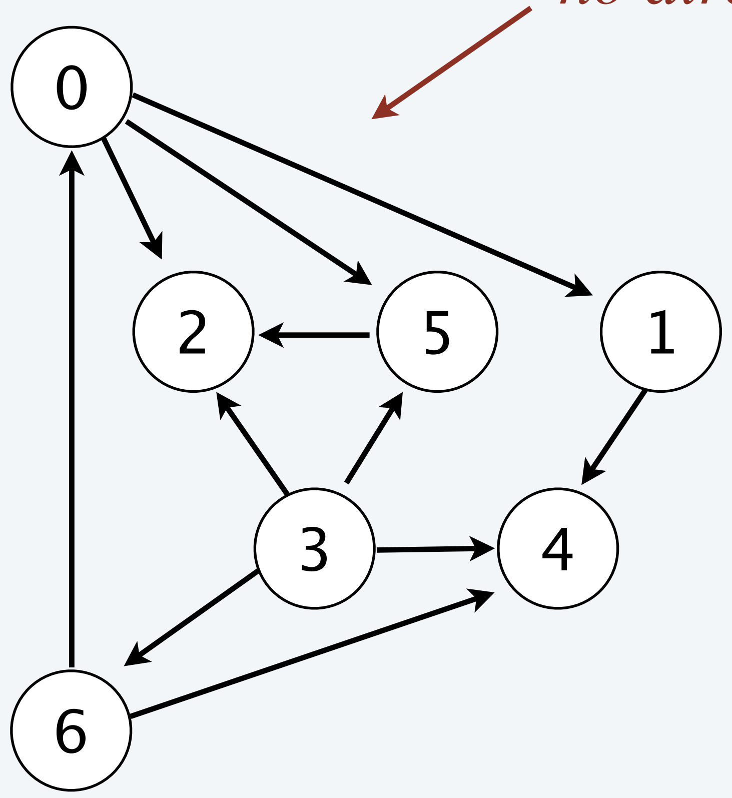
Topological sort. Given a DAG, find a linear ordering of the vertices so that for every edge $v \to w$, v comes before w in the ordering.

edges in DAG define a "partial order" for vertices

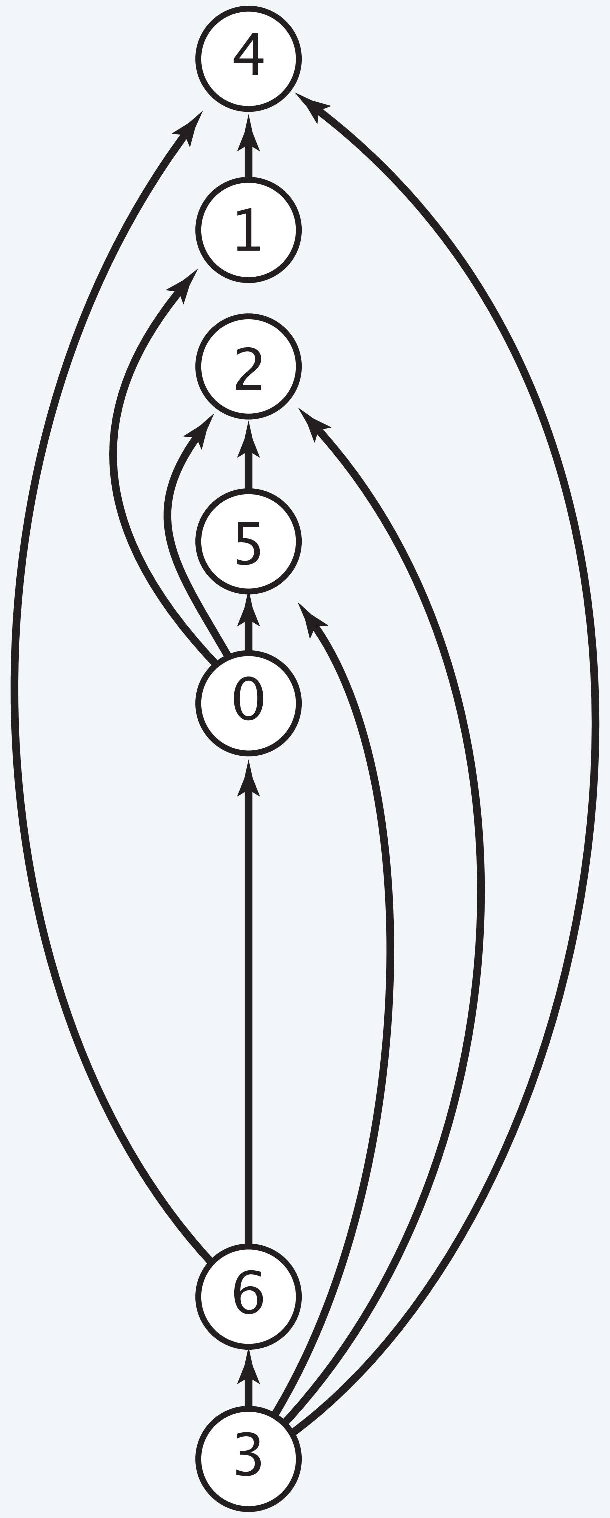
0→5	0→2
0→1	3→6
3→5	3→4
5→2	6→4
6→0	3→2
1→4	

directed edges

no directed cycles



DAG



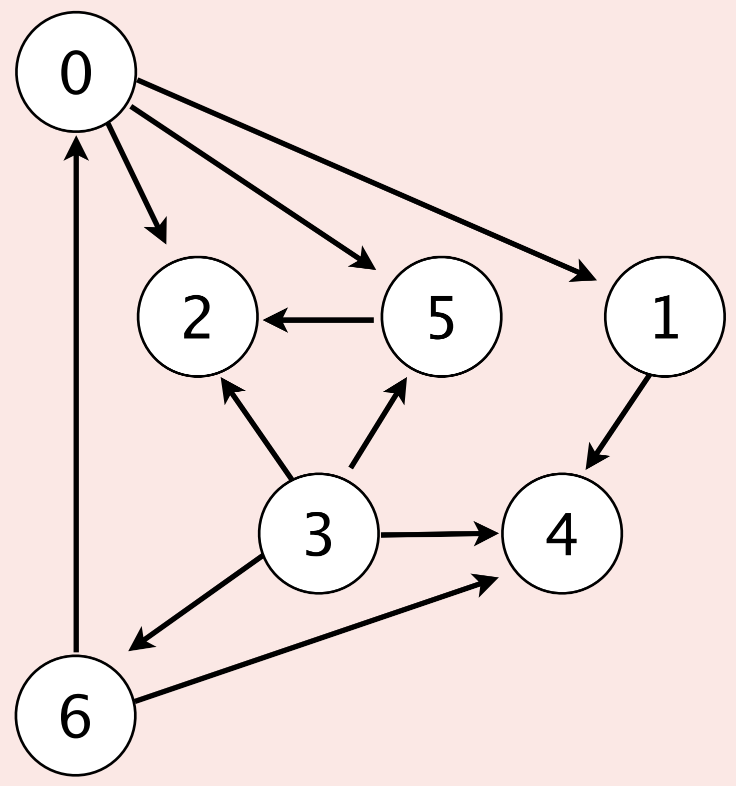
topological ordering: 3 6 0 5 2 1 4

Graphs and digraphs II: poll 3

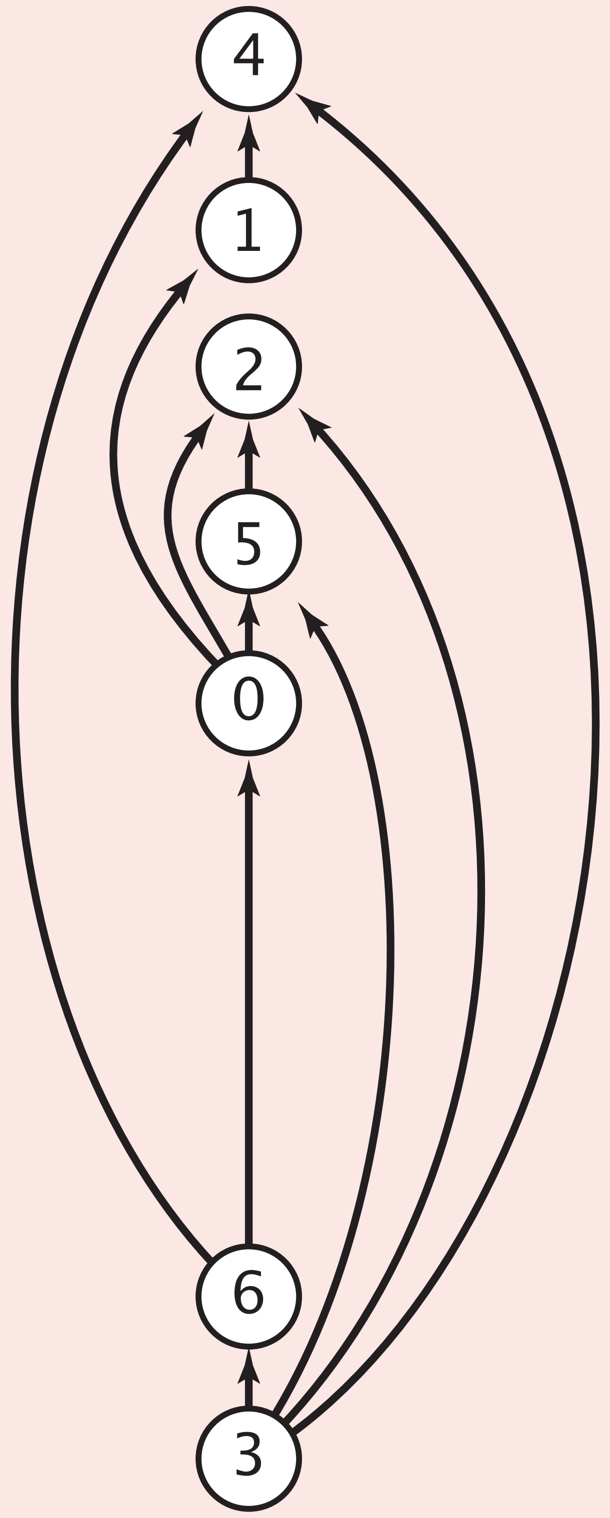
Suppose that you want to topologically sort the vertices in a DAG.

Which graph-search algorithm should you use?

- **A.** Depth-first search.
- **B.** Breadth-first search.
- **C.** Either A or B.
- **D.** Neither A nor B.



DAG

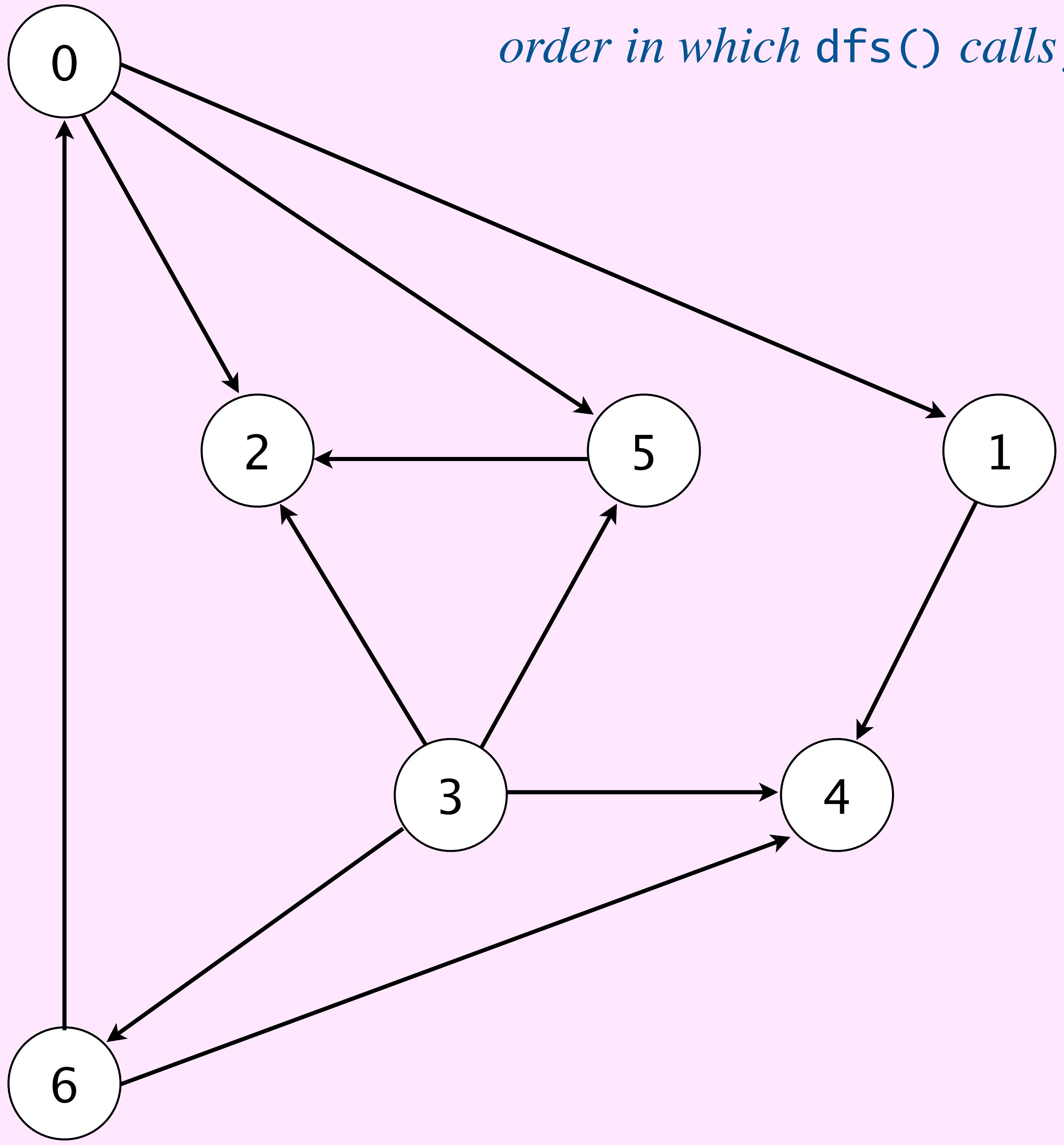


topological ordering: 3 5 0 5 2 1 4

Topological sort demo

- Run depth-first search.
- Return vertices in reverse DFS postorder.

order in which `dfs()` *calls finish*



a directed acyclic graph

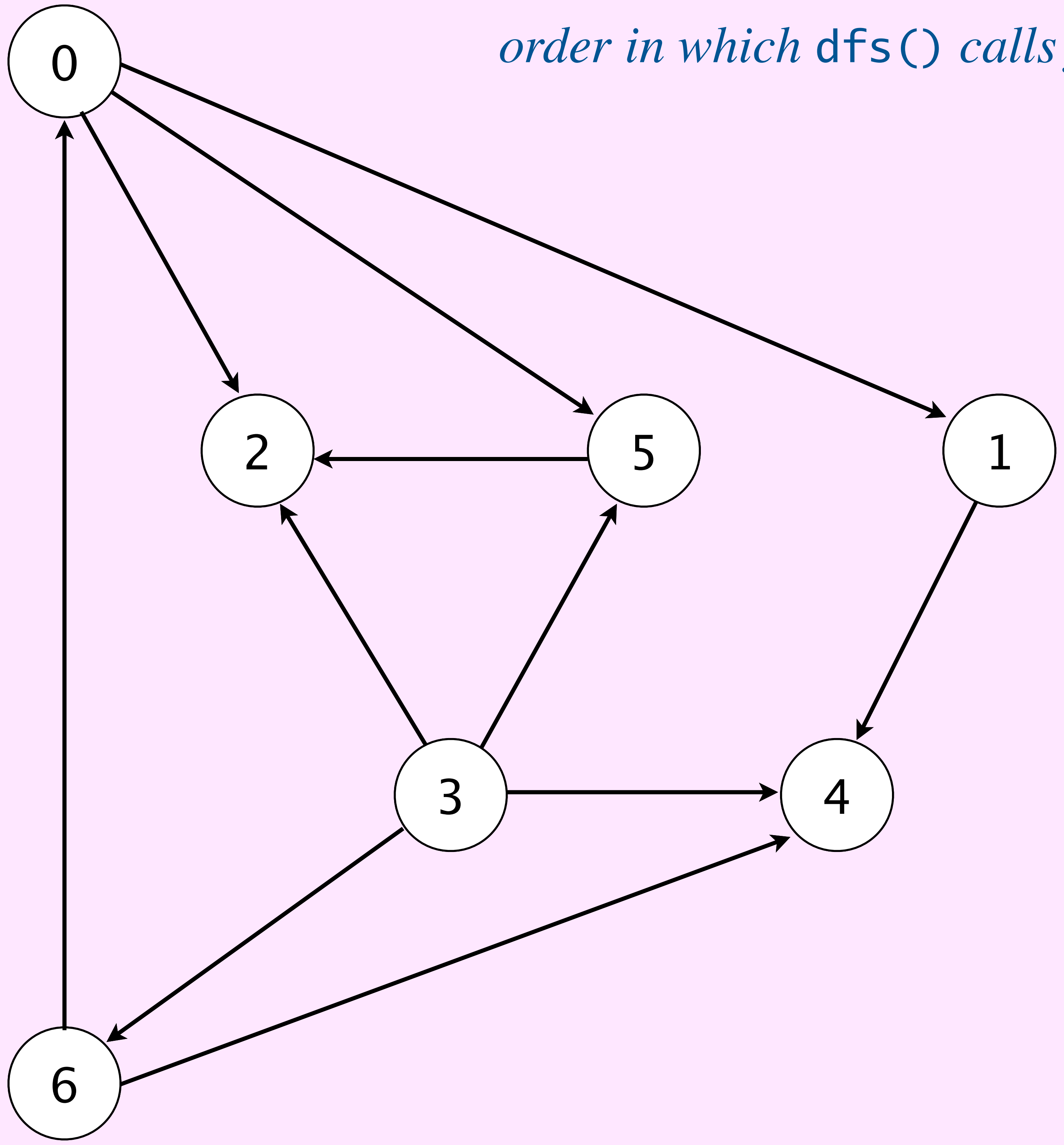
tinyDAG7.txt

```
7
11
 0 5
 0 2
 0 1
 3 6
 3 5
 3 4
 5 2
 6 4
 6 0
 3 2
```

Topological sort demo

- Run depth-first search.
- Return vertices in reverse DFS postorder.

order in which `dfs()` *calls finish*



DFS postorder

4 1 2 5 0 6 3

topological ordering
(reverse DFS postorder)

3 6 0 5 2 1 4

done

Depth-first search: reverse postorder

```
public class DepthFirstOrder {
   private boolean[] marked;
   private Stack<Integer> reversePostorder;

   public DepthFirstOrder(Digraph G) {
      reversePostorder = new Stack<>();
      marked = new boolean[G.V()];
      for (int v = 0; v < G.V(); v++)
         if (!marked[v])
            dfs(G, v);
   }

   private void dfs(Digraph G, int v) {
      marked[v] = true;
      for (int w : G.adj(v))
         if (!marked[w]) dfs(G, w);
      reversePostorder.push(v);
   }

   public Iterable<Integer> reversePostorder() {
      return reversePostorder;
   }

}
```

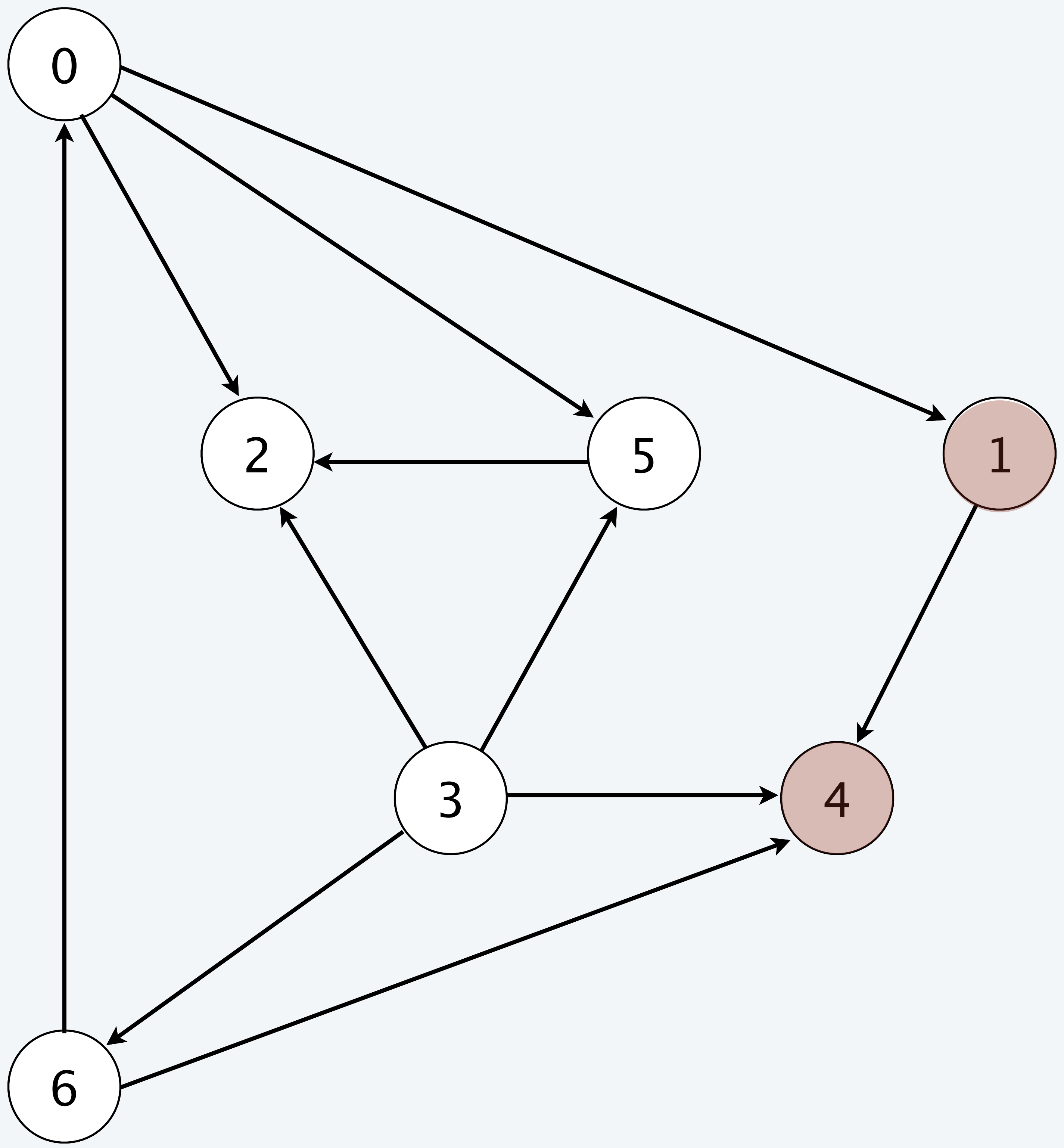
run DFS from all vertices

return vertices in reverse DFS postorder

Topological sort in a DAG: intuition

Why is the reverse DFS postorder of a DAG a topological order?

- First vertex in DFS postorder has outdegree 0.
- Second vertex in DFS postorder can point only to first vertex.
- …



DFS postorder

4 1 2 5 0 6 3

topological ordering (reverse DFS postorder)

3 6 0 5 2 1 4

Topological sort in a DAG: proof of correctness

Proposition. Reverse DFS postorder of a DAG is a topological order.

Pf. Consider any edge $v \to w$. When `dfs(v)` is called:

- Case 1: `dfs(w)` has already been called and returned.
  - thus, w appears before v in DFS postorder

- Case 2: `dfs(w)` has not yet been called.
  - `dfs(w)` will get called directly or indirectly by `dfs(v)`
  - so, `dfs(w)` will return before `dfs(v)` returns
  - thus, w appears before v in DFS postorder

- Case 3: `dfs(w)` has already been called, but has not yet returned.
  - function-call stack contains directed path from w to v
  - appending edge $v \to w$ to this path yields a directed cycle
  - contradiction (it's a DAG)

```
dfs(0)
   dfs(1)
      dfs(4)
      4 done
   1 done
   dfs(2)
   2 done
   dfs(5)
      check 2
   5 done
0 done
check 1
check 2
dfs(3)
   check 2
   check 4
   check 5
   dfs(6)
      check 0
      check 4
   6 done
3 done
check 4
check 5
check 6
done
```

$v = 3$

case 1 $(w = 2, 4, 5)$

case 2 $(w = 6)$

Topological sort in a DAG: running time

Proposition. For any DAG, the DFS-based algorithm computes a topological order in $\Theta(E + V)$ time.

Pf. For every vertex v, there is exactly one call to `dfs(v)`.

critical that vertices are marked (and never unmarked)

Q. What if we run the algorithm on a digraph that is not a DAG?

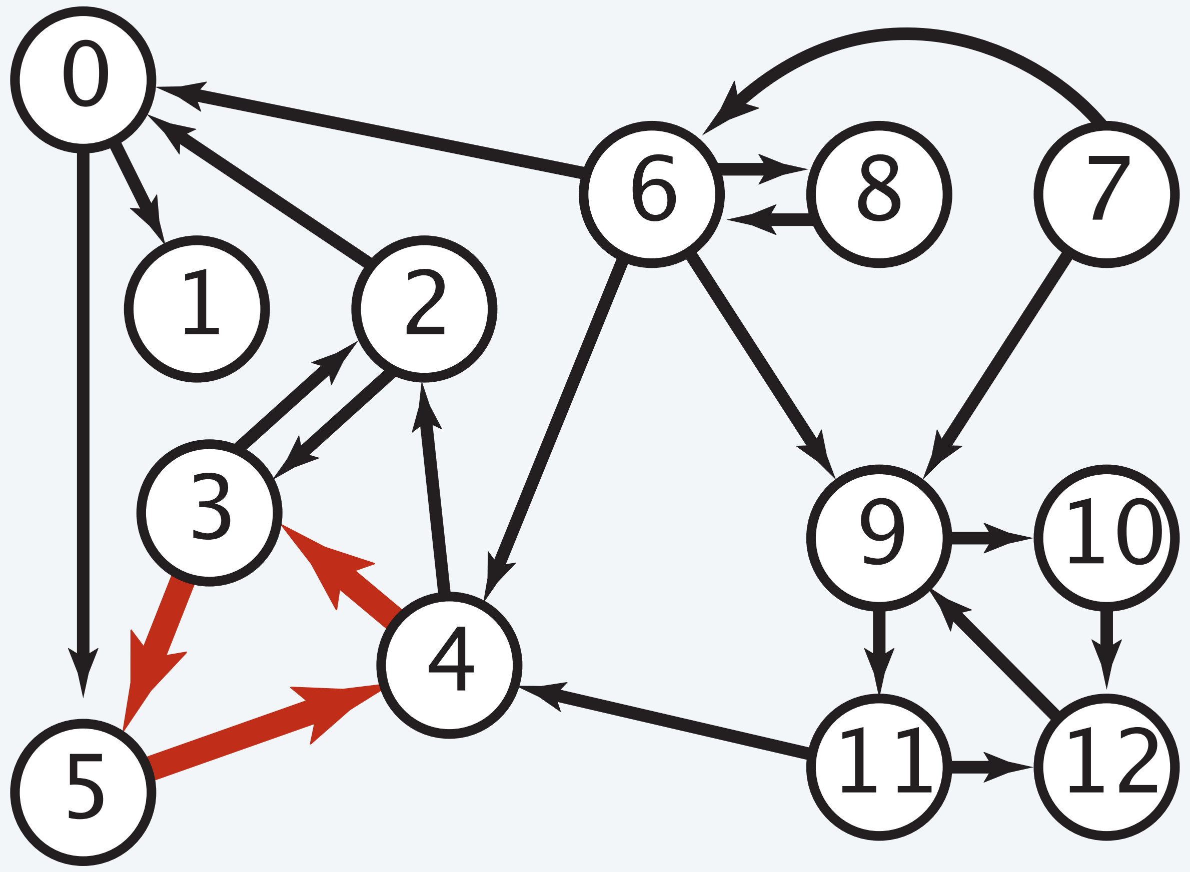
A. Reverse DFS postorder is still well defined, but it won't be a topological order.

Directed cycle detection

Proposition. A digraph has a topological order if and only if contains no directed cycle.

Pf.

- Directed cycle $\Longrightarrow$ no topologic order possible (consider vertices in the cycle).
- No directed cycle $\Longrightarrow$ reverse DFS postorder is a topological order.



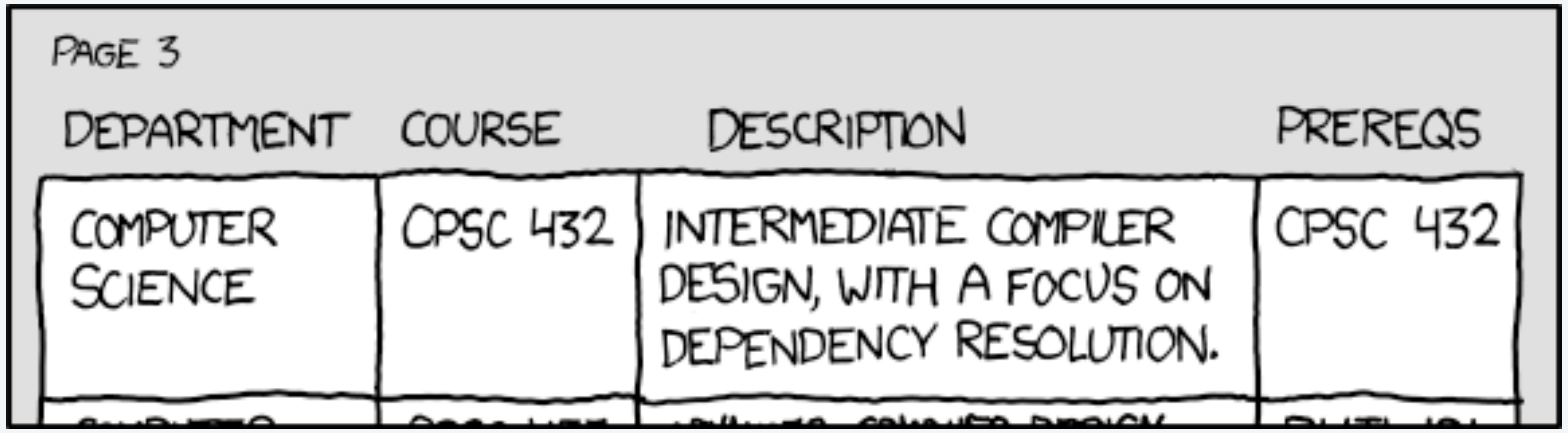
a digraph with a directed cycle

Goal. Given a digraph, find a directed cycle (if one exists).

Solution. DFS. What else? See textbook/precept.

Directed cycle detection application: precedence scheduling

Scheduling. Given a set of tasks to be completed with precedence constraints, in which order should we schedule the tasks?



https://xkcd.com/754

Remark. A directed cycle implies scheduling problem is infeasible.

Directed cycle detection application: cyclic inheritance

The Java compiler does directed cycle detection.

```
public class A extends B {
   ...
}
```

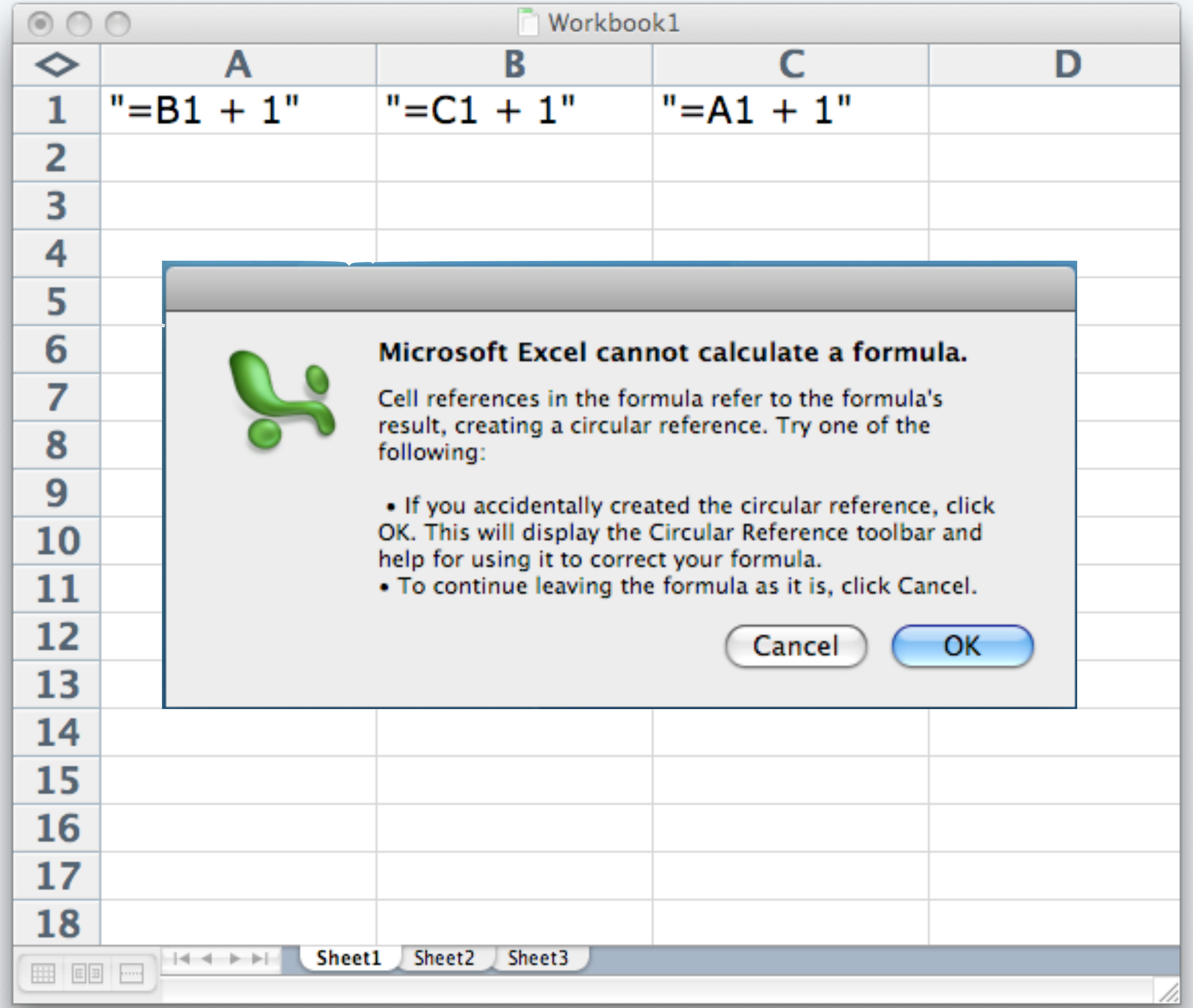
```
public class B extends C {
   ...
}
```

```
public class C extends A {
   ...
}
```

```
~/cos226/graph> javac A.java
A.java:1: cyclic inheritance involving A
public class A extends B { }
       ^
1 error
```

Directed cycle detection application: spreadsheet recalculation

Microsoft Excel does directed cycle detection.



Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

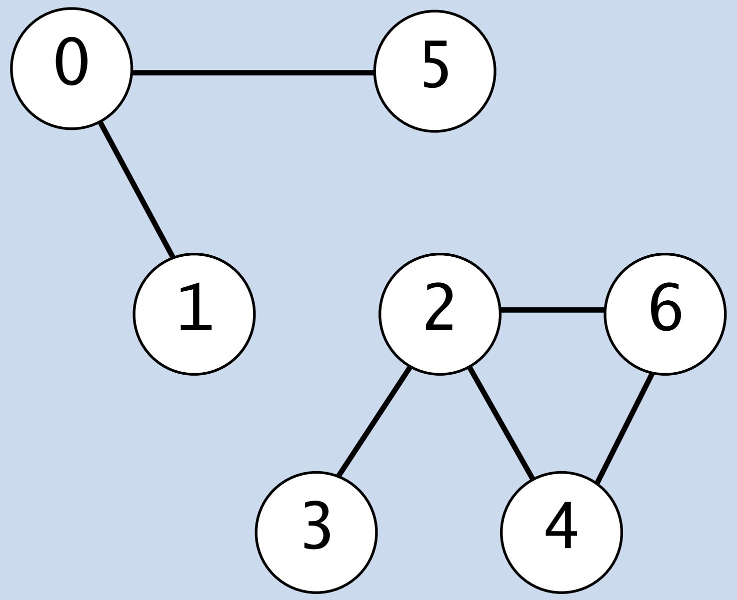
https://algs4.cs.princeton.edu

4. GRAPHS AND DIGRAPHS II

- *breadth-first search (in directed graphs)*
- *breadth-first search (in undirected graphs)*
- *topological sort*
- *challenges*

Graph-processing challenge 1

Problem. Identify connected components.



0-1
0-5
2-6
2-3
2-4
4-6

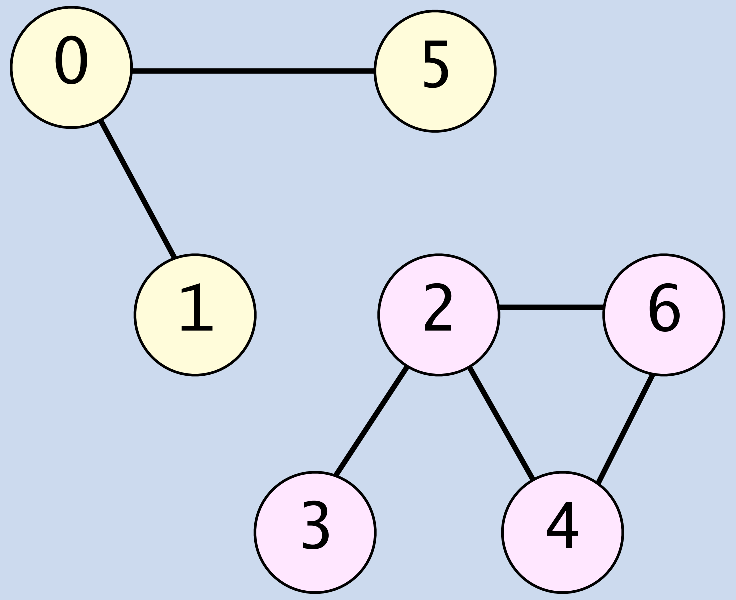
How difficult?

A. Diligent algorithms student could do it.

B. Hire an expert.

C. Intractable.

D. No one knows.



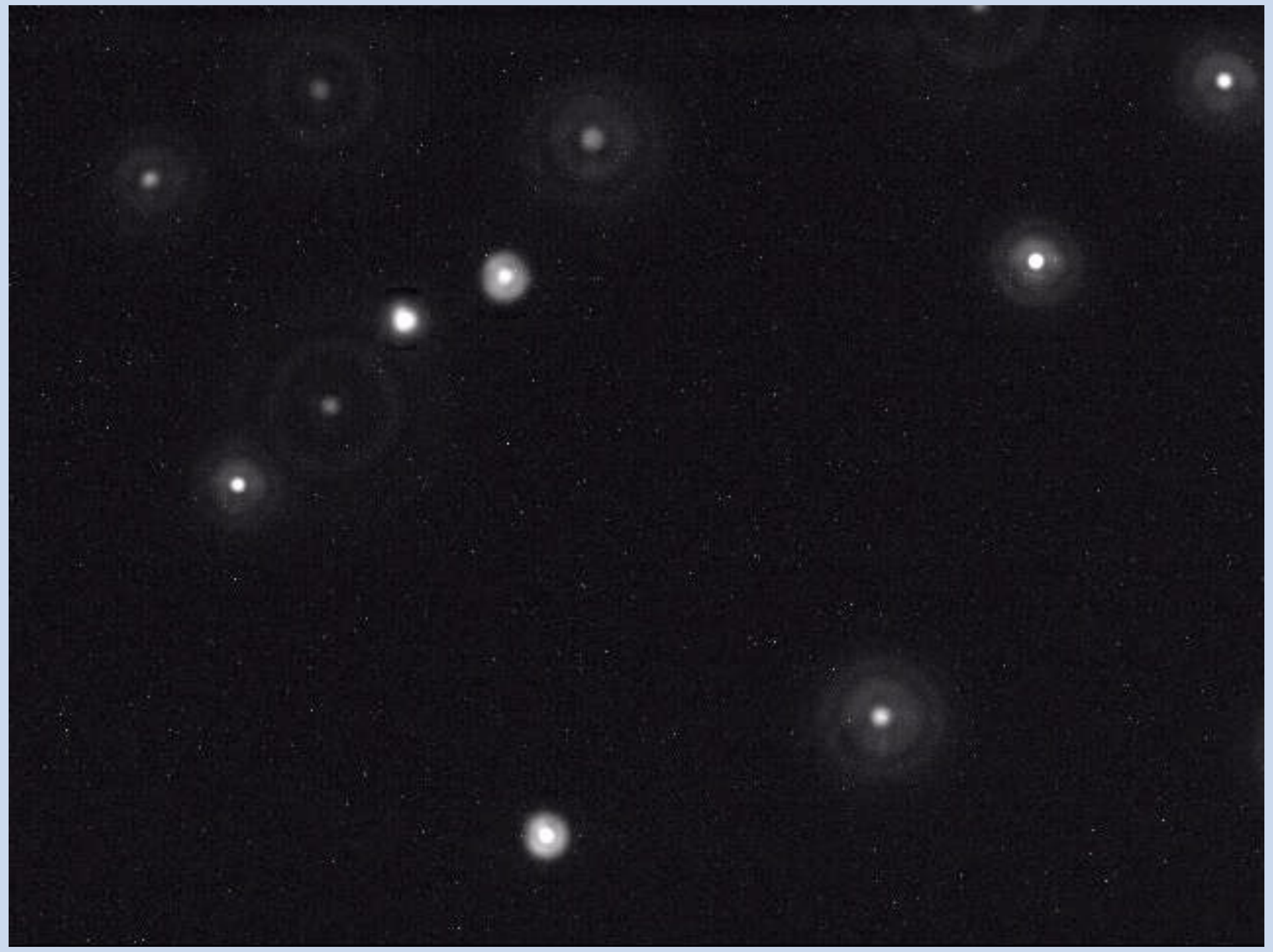
v	id[]
0	0
1	0
2	1
3	1
4	1
5	0
6	1

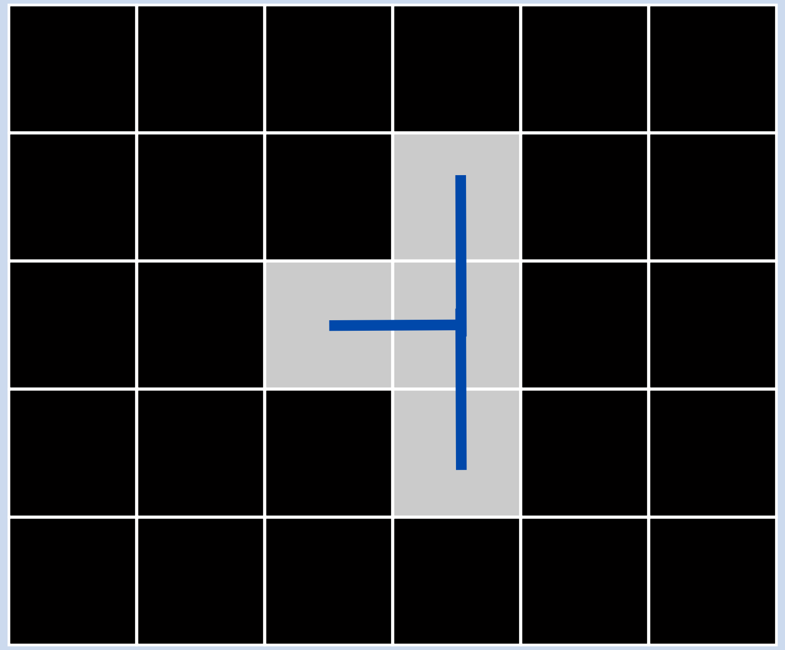
Graph-processing challenge 1

Problem. Identify connected components.

Particle detection. Given grayscale image of particles, identify "blobs."

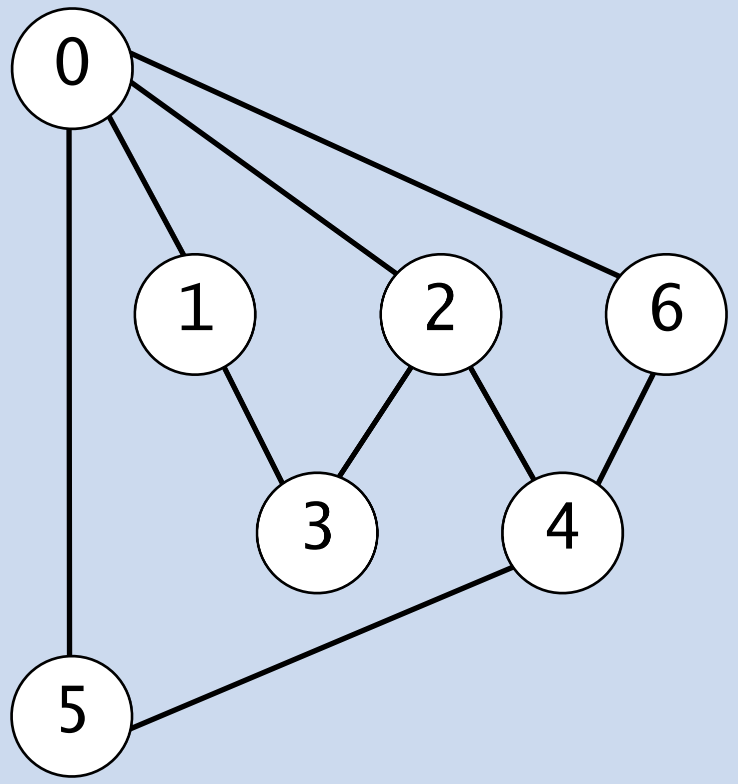
- Vertex: pixel.
- Edge: between two adjacent pixels with grayscale value ≥ 70.
- Blob: connected component of 20–30 pixels.





Graph-processing challenge 2

Problem. Is a graph bipartite?



0-1
0-2
0-5
0-6
1-3
2-3
2-4
4-5
4-6

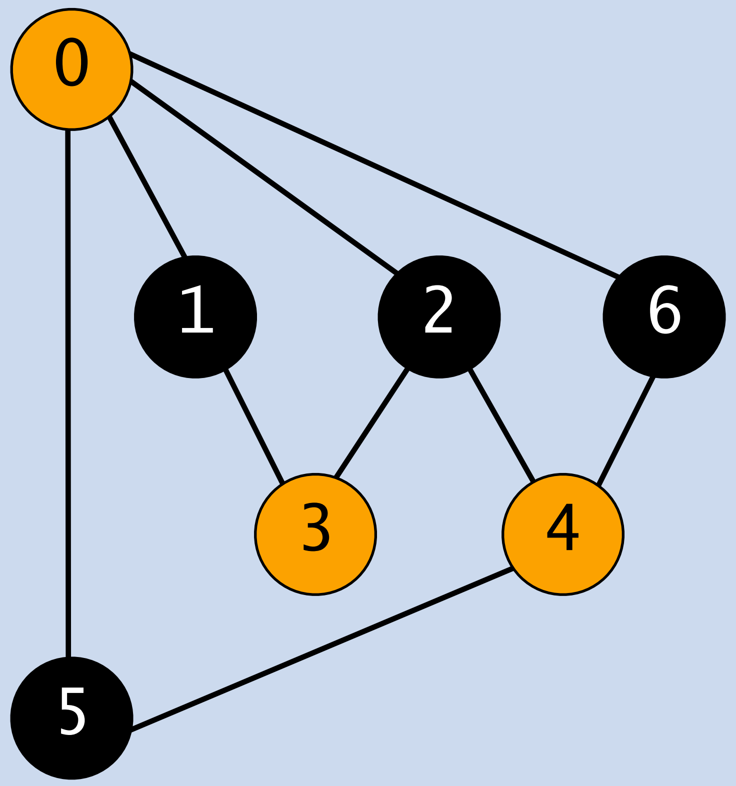
How difficult?

A. Diligent algorithms student could do it.

B. Hire an expert.

C. Intractable.

D. No one knows.

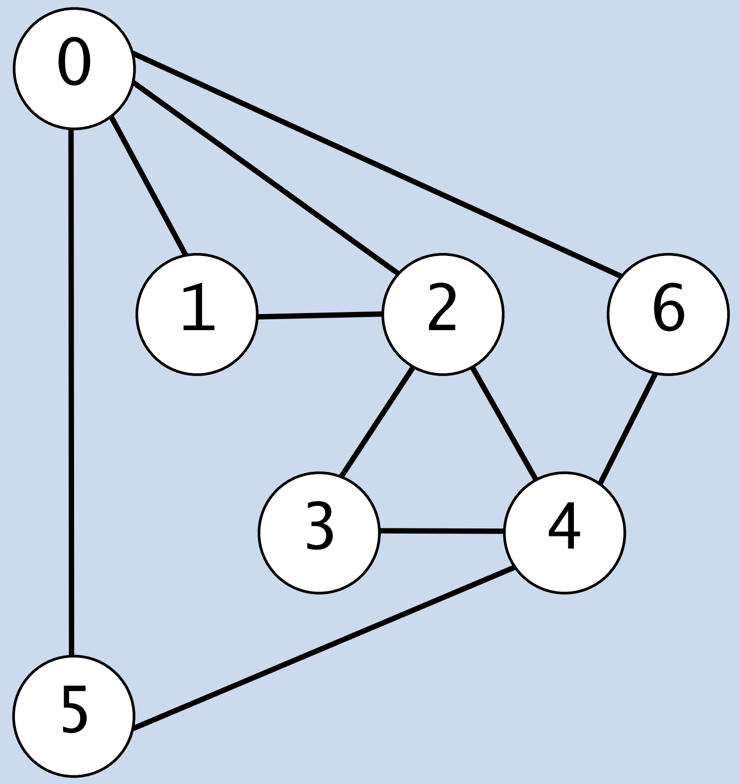


0-1
0-2
0-5
0-6
1-3
2-3
2-4
4-5
4-6

{ 1, 2, 5, 6 }

Graph-processing challenge 3

Problem. Is there a (non-simple) cycle that uses every edge exactly once?



0-1
0-2
0-5
0-6
1-2
2-3
2-4
3-4
4-5
4-6

0-1-2-3-4-2-0-6-4-5-0

How difficult?

A. Diligent algorithms student could do it.

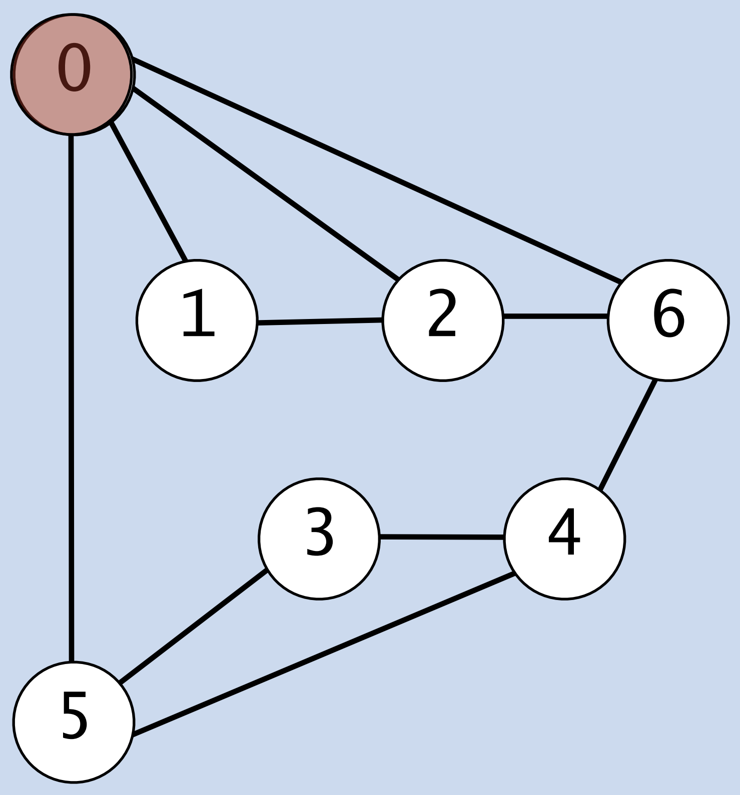
B. Hire an expert.

C. Intractable.

D. No one knows.

Graph-processing challenge 4

Problem. Is there a cycle that uses every vertex exactly once?



0-1
0-2
0-5
0-6
1-2
2-6
3-4
3-5
4-5
4-6

0-5-3-4-6-2-1-0

How difficult?

A. Diligent algorithms student could do it.

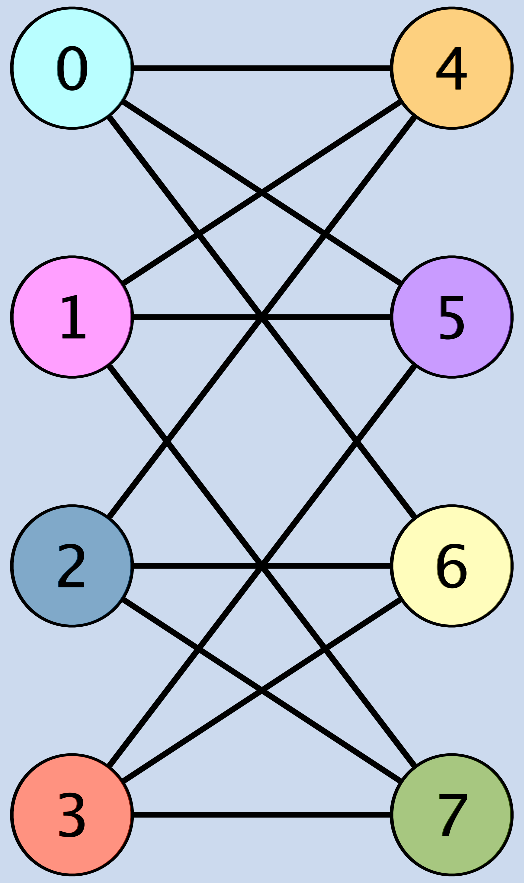
B. Hire an expert.

C. Intractable.

D. No one knows.

Graph-processing challenge 5

Problem. Are two graphs identical except for vertex names?



G_1

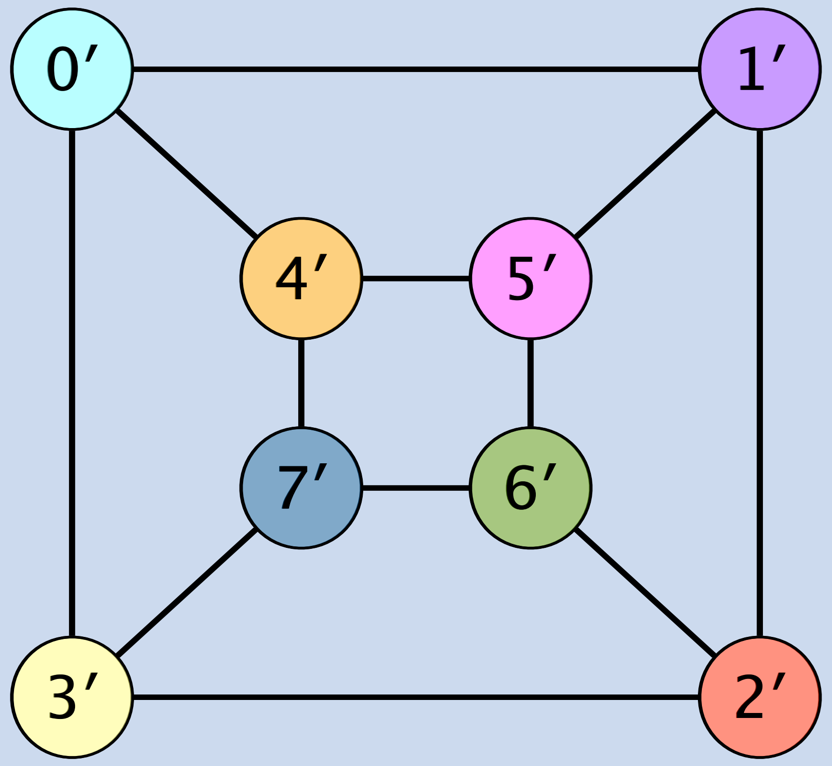
How difficult?

A. Diligent algorithms student could do it.

B. Hire an expert.

C. Intractable.

D. No one knows.



G_2

$f(0) = 0'$

$f(1) = 5'$

$f(2) = 7'$

$f(3) = 2'$

$f(4) = 4'$

$f(5) = 1'$

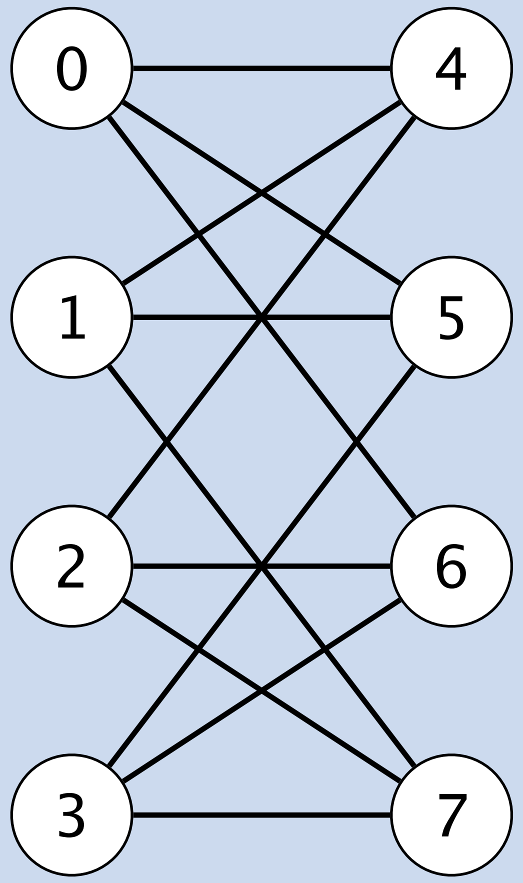
$f(6) = 3'$

$f(7) = 6'$

Graph-processing challenge 6

Problem. Can you draw a graph in the plane with no crossing edges?

try it yourself at
`https://www.jasondavies.com/planarity`



0-1	2-4
0-5	2-6
0-6	2-7
1-4	3-5
1-5	3-6
1-7	3-7

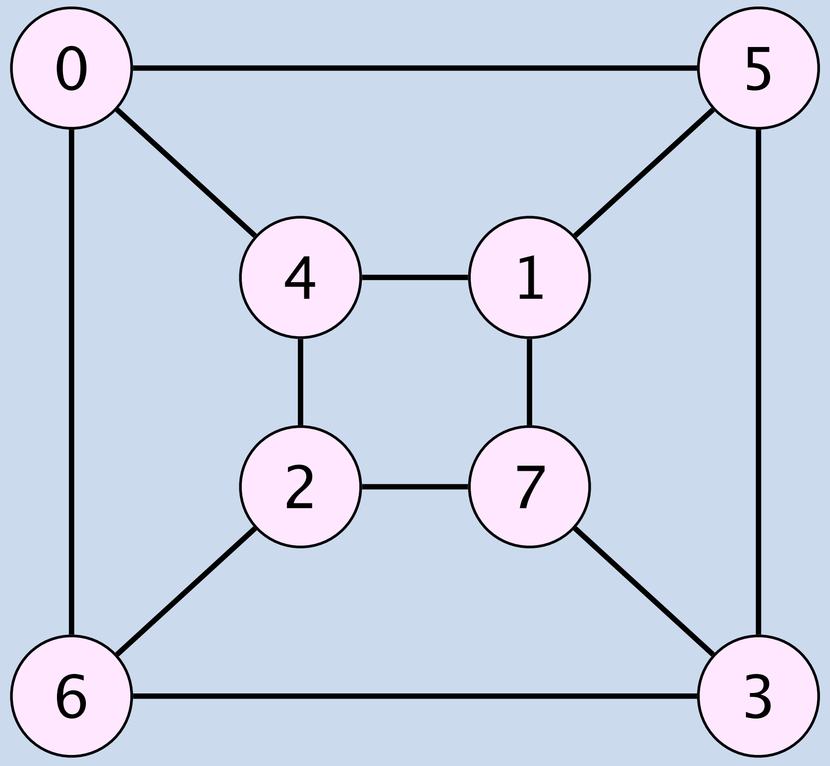
How difficult?

A. Diligent algorithms student could do it.

B. Hire an expert.

C. Intractable.

D. No one knows



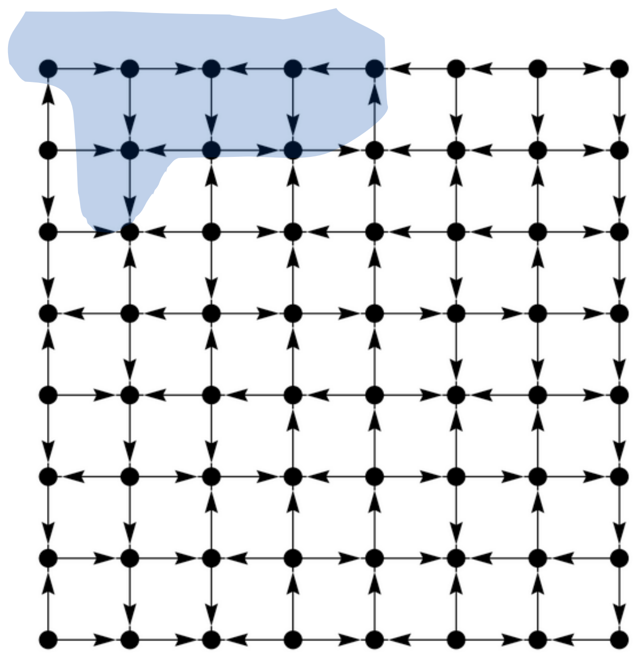
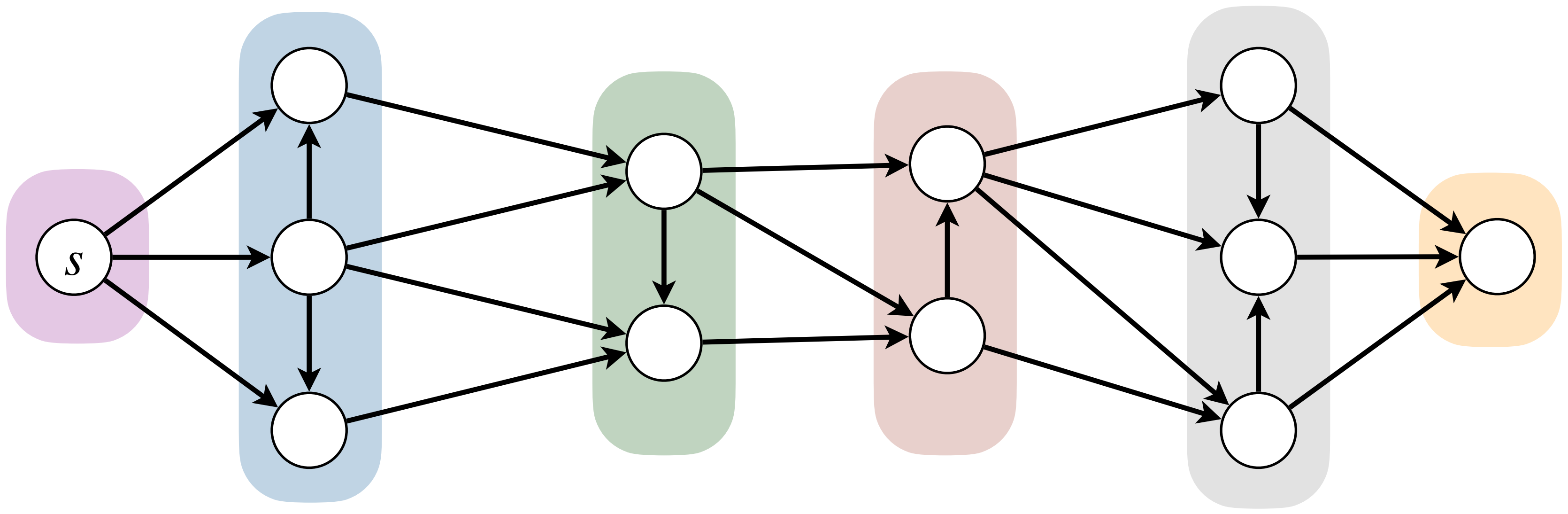
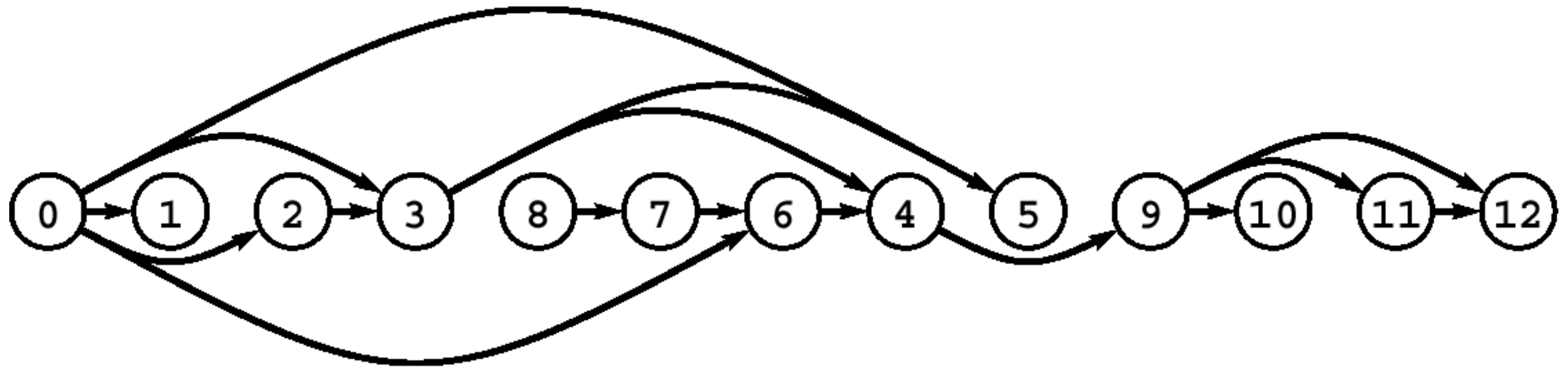
yes (a planar embedding)

Graph processing summary

BFS and DFS enables efficient solution of many (but not all) graph and digraph problems.

	graph problem	BFS	DFS	time
😁	s–t path	✔	✔	$E + V$
😁	shortest s–t path	✔		$E + V$
😕	shortest directed cycle	✔		$E\,V$
😁	Euler cycle		✔	$E + V$
👿	Hamilton cycle			$2^{1.657\,V}$
😁	bipartiteness (odd cycle)	✔	✔	$E + V$
😁	connected components	✔	✔	$E + V$
😁	strong components		✔	$E + V$
😀	planarity		✔	$E + V$
🤷🏽‍♀️	graph isomorphism			$2^{c \ln^3 V}$

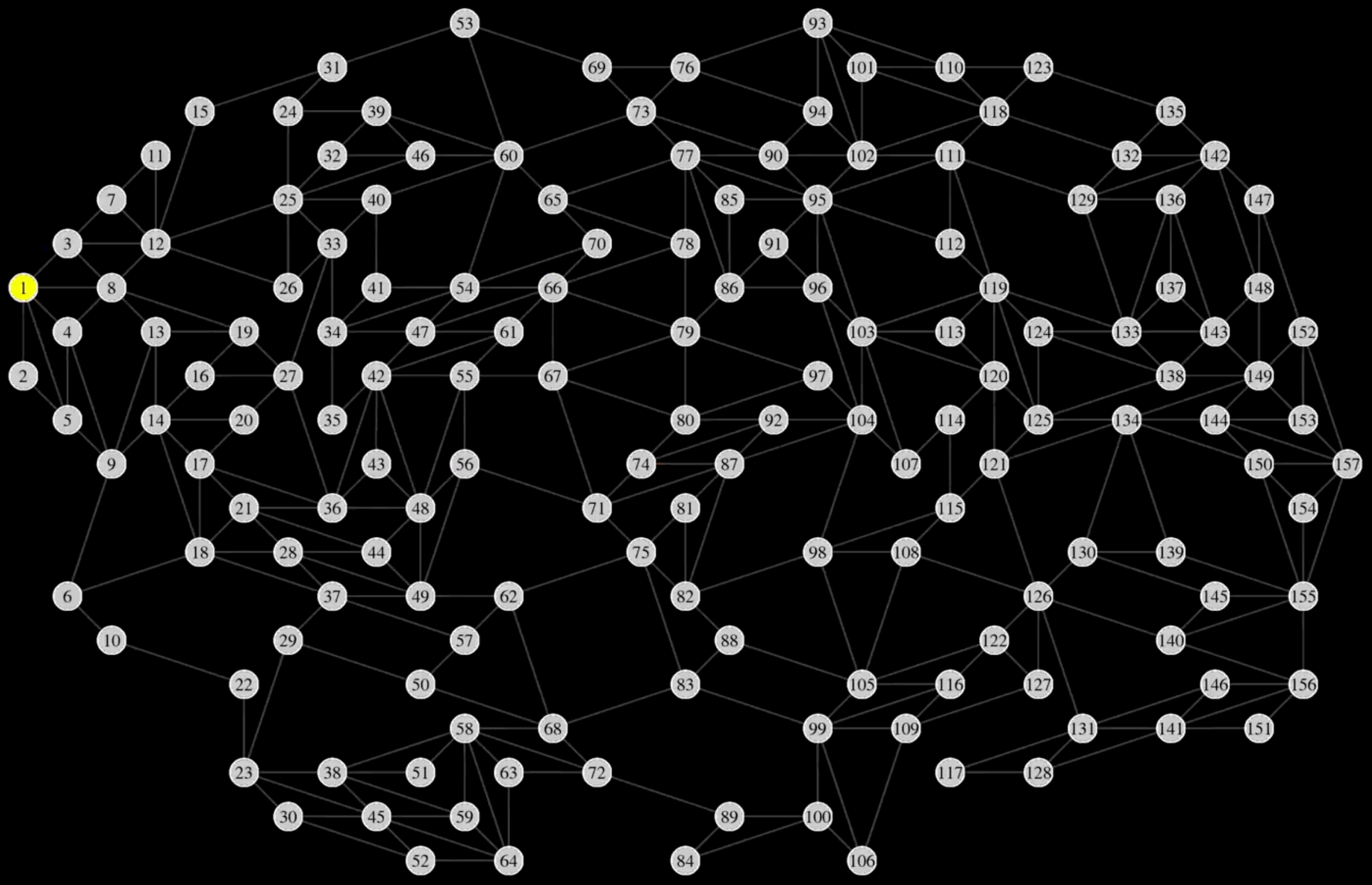
Graph-processing summary: algorithms of the week

single-source reachability		DFS/BFS
shortest paths		BFS
topological sort		DFS

Credits

media	source	license
ARPANET	Wikimedia	CC BY-SA 4.0
Oracle of Bacon	oracleofbacon.org	
Kevin Bacon Game	Endless Games	
Six Degrees of Hollywood	Paradox Apps	
Pedigree of King Charles II	Waterford Treasures	
Habsburg Coat of Arms	Wikimedia	CC BY-SA 3.0
Bayesian Network	Thornley et. al	
Dependencies	xkcd	CC BY-NC 2.5
Brownian Motion	William Ryu	permission by author
BFS Graph Visualization	Gerry Jenkins	

BFS visualization (by Gerry Jenkins)



https://www.youtube.com/watch?v=x-VTfcmrLEQ