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3.2 Binary Search Trees

- *BSTs*
- *ordered operations*
- *iteration*

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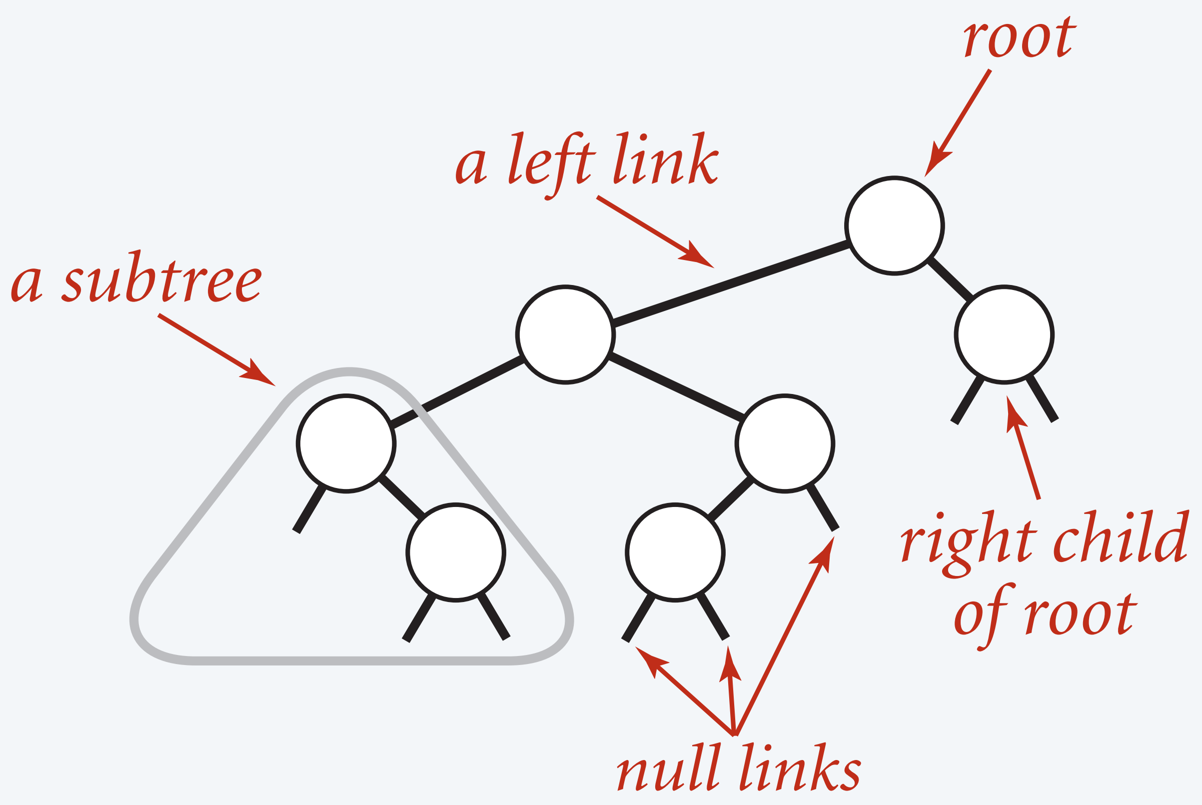
- *ordered operations*
- *iteration*

Binary search trees

Definition. A BST is a binary tree in symmetric order.

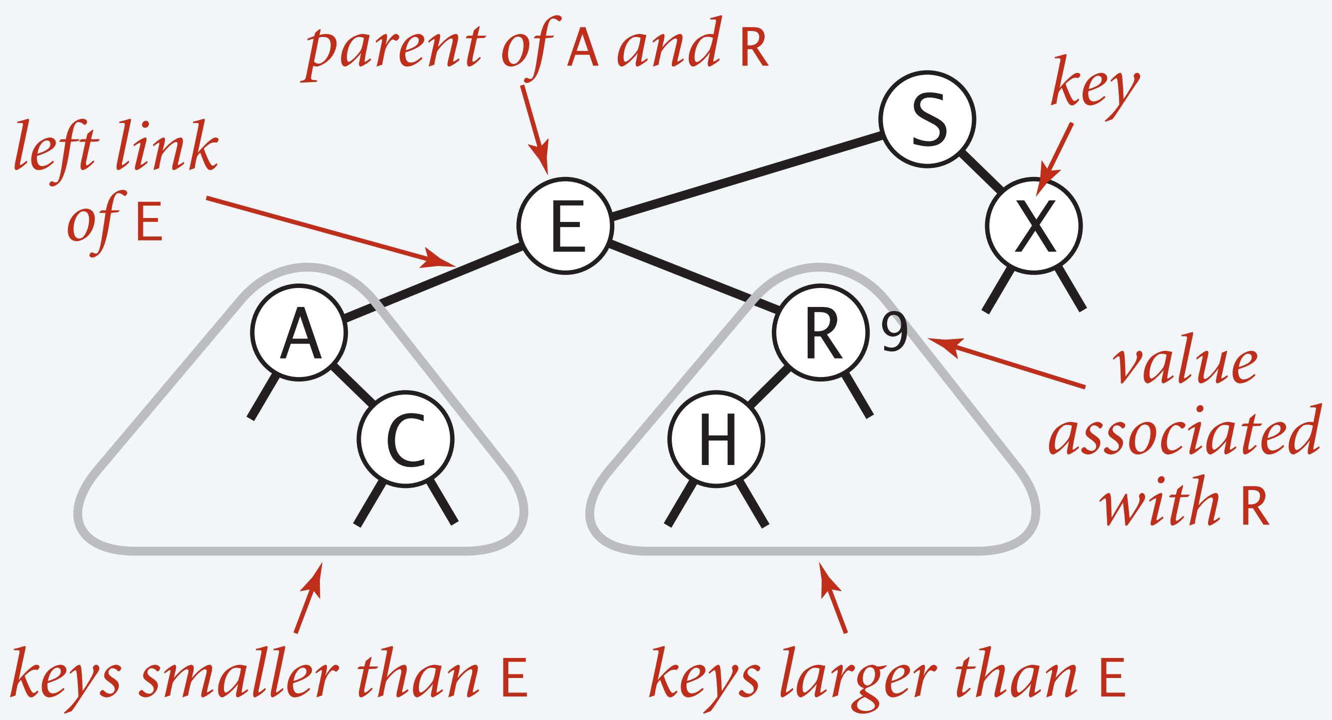
A binary tree is either:

- Empty.
- A node with links to two disjoint binary trees—the left subtree and the right subtree.



Symmetric order. Each node has a key that is:

- Strictly larger than all keys in its left subtree.
- Strictly smaller than all keys in its right subtree.
- [Duplicate keys not permitted.]



Which of the following properties hold?

A. If a binary tree is max-heap ordered, then it is symmetrically ordered.

B. If a binary tree is symmetrically ordered, then it is max-heap ordered.

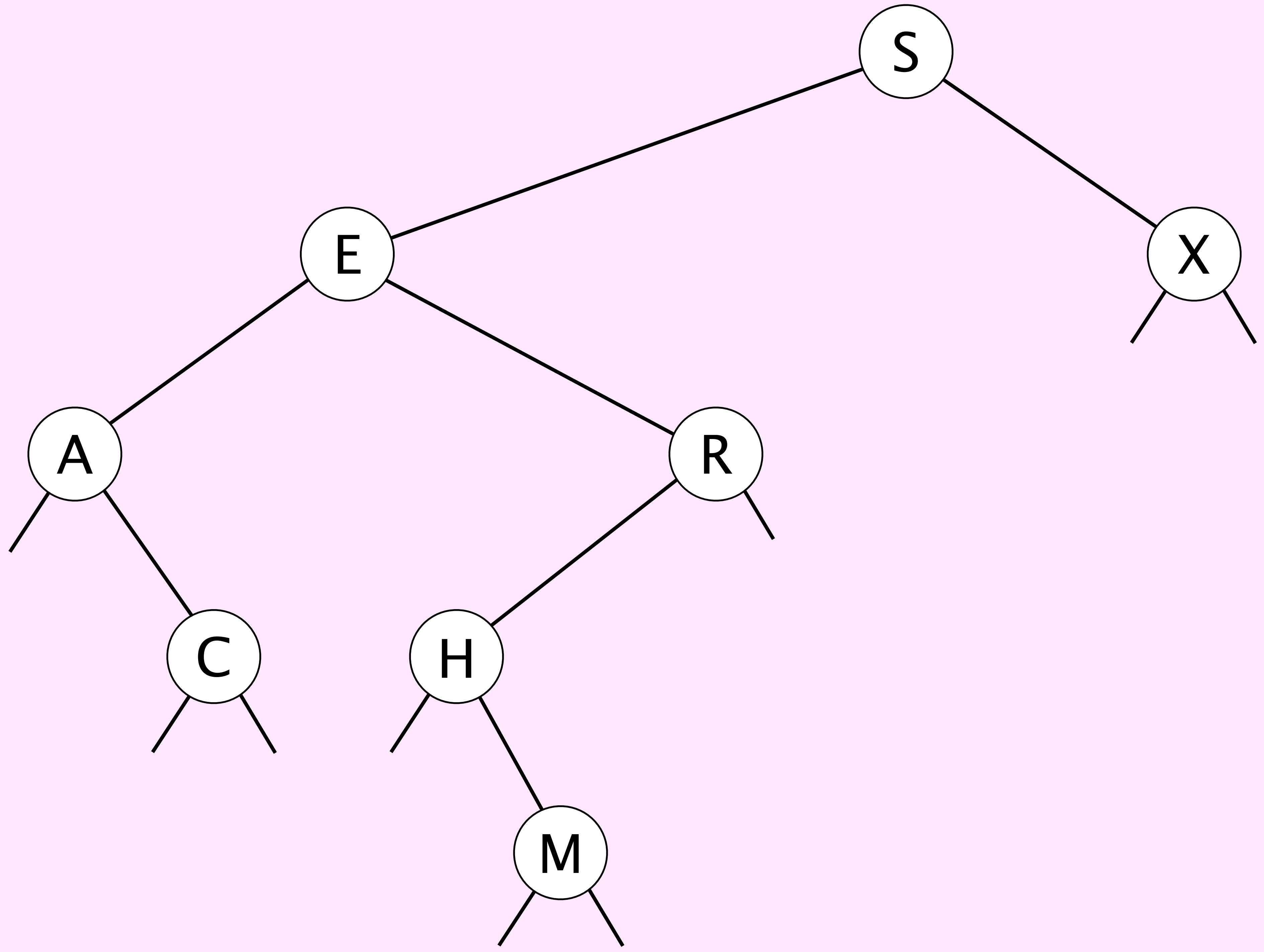
C. Both A and B.

D. Neither A nor B.

Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

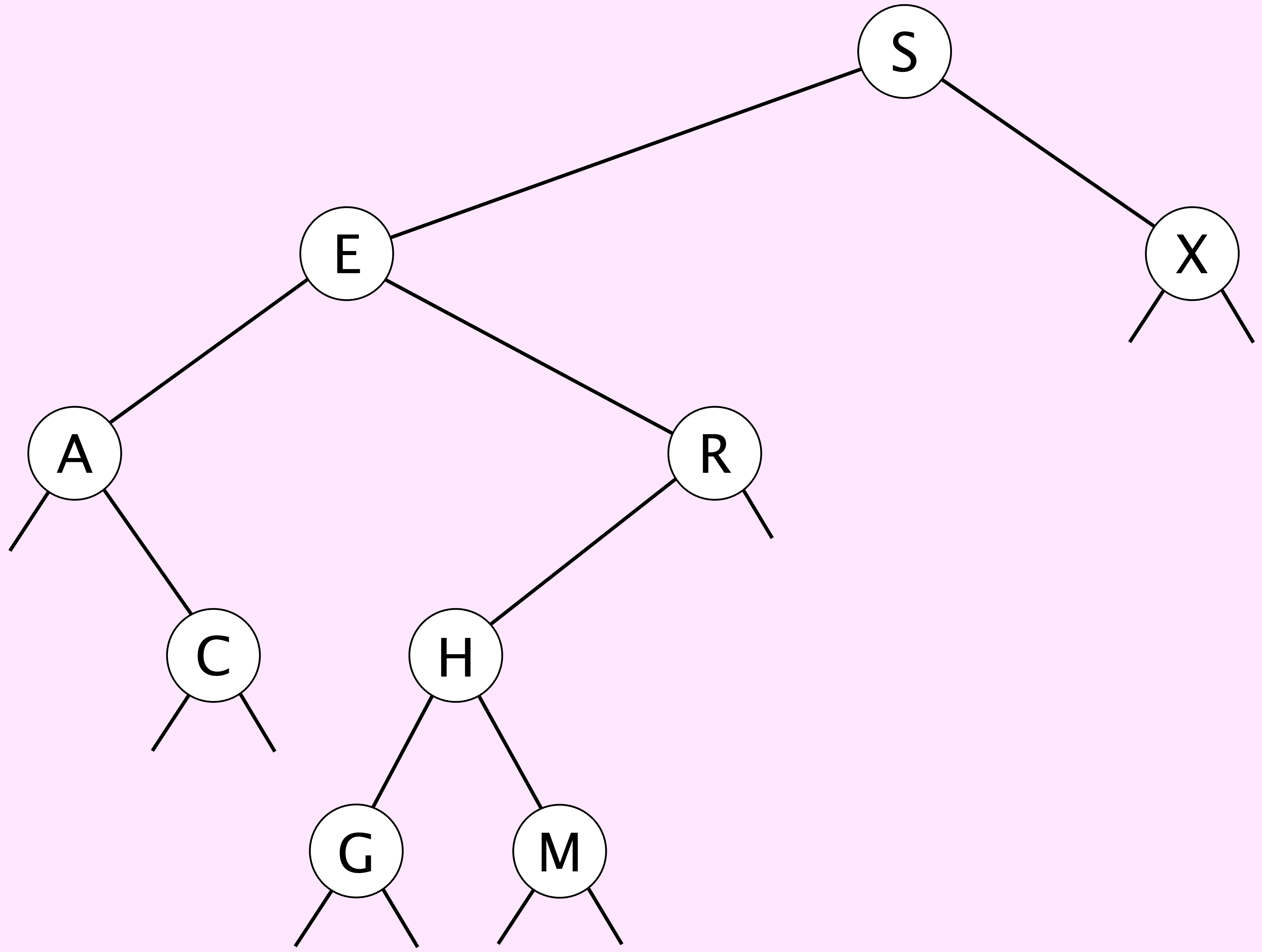
successful search for H



Binary search tree demo

Insert. If less, go left; if greater, go right; if `null`, insert.

insert G



BST representation in Java

Java representation. A BST holds a reference to a root Node.

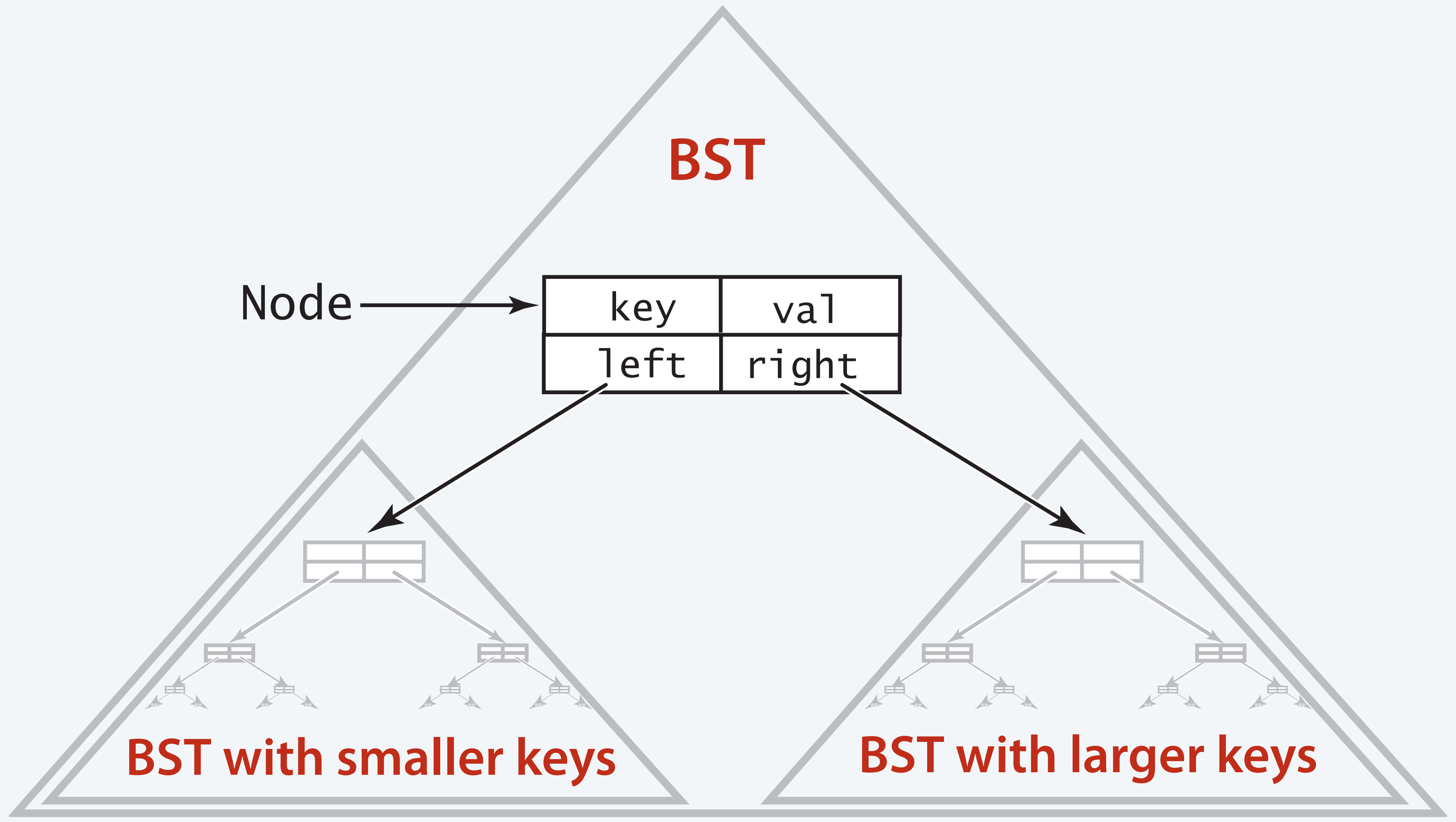
A Node is composed of four fields:

- A Key and a Value.
- A reference to the left and right subtree.

smaller keys *larger keys*

```java
private class Node {
   private Key key;
   private Value val;
   private Node left, right;

   public Node(Key key, Value val) {
      this.key = key;
      this.val = val;
   }
}
```

Key and Value are generic types; Key is Comparable



binary search tree

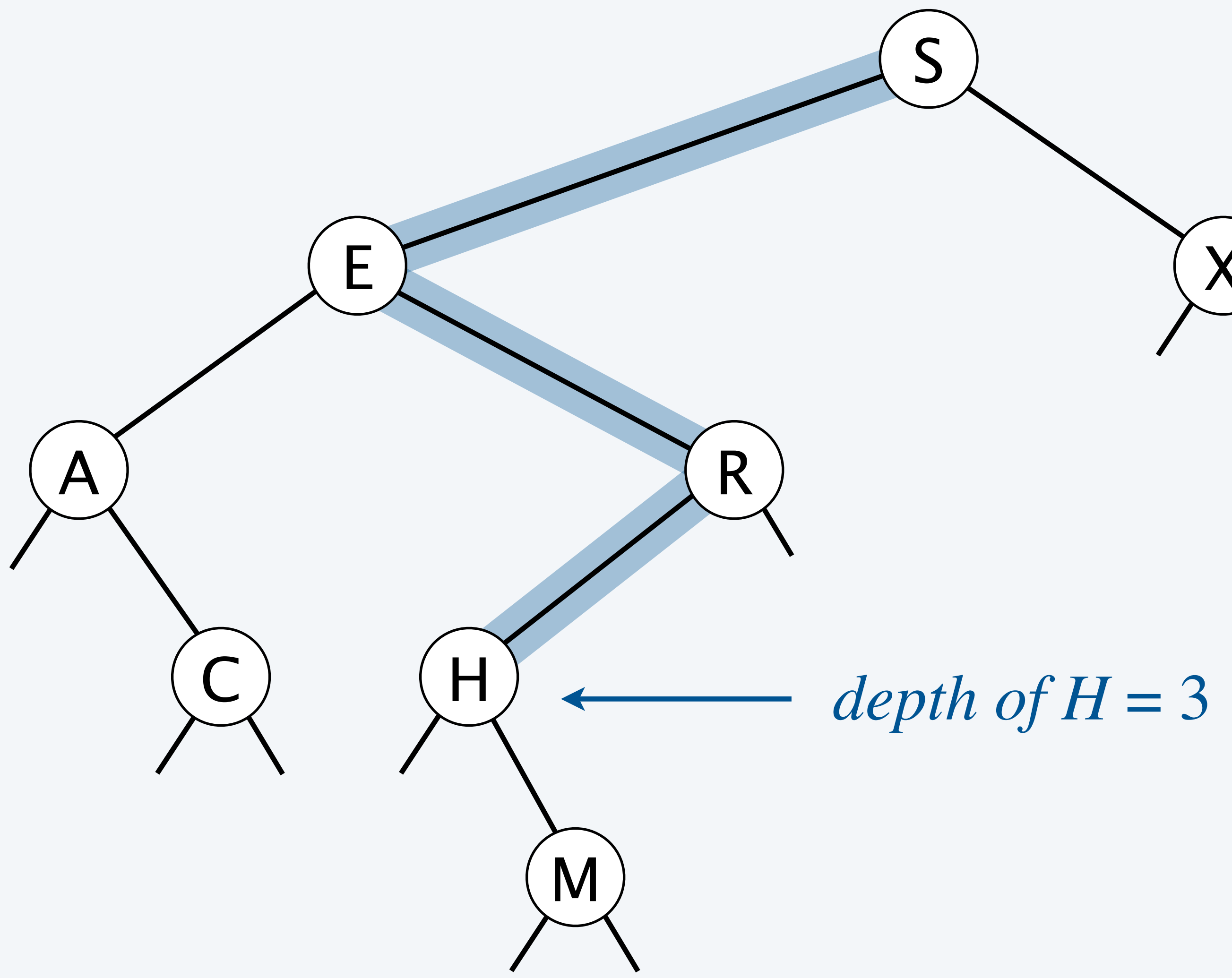
BST implementation (skeleton)

```
public class BST<Key extends Comparable<Key>, Value> {

    private Node root;        ← root of BST

    private class Node
    {  /* see previous slide */  }

    public void put(Key key, Value val)
    {  /* see slide in this section */  }

    public Value get(Key key)
    {  /* see next slide */  }

    public Iterable<Key> keys()
    {  /* see slides in next section */  }

    public void delete(Key key)
    {  /* see textbook */  }

}
```

BST search: Java implementation

Get. Return value corresponding to given key, or `null` if no such key.

```
public Value get(Key key) {
   Node x = root;
   while (x != null) {
      int cmp = key.compareTo(x.key);
      if      (cmp < 0) x = x.left;
      else if (cmp > 0) x = x.right;
      else return x.val;
   }
   return null;
}
```



Cost. Number of compares = 1 + depth of deepest node reached.

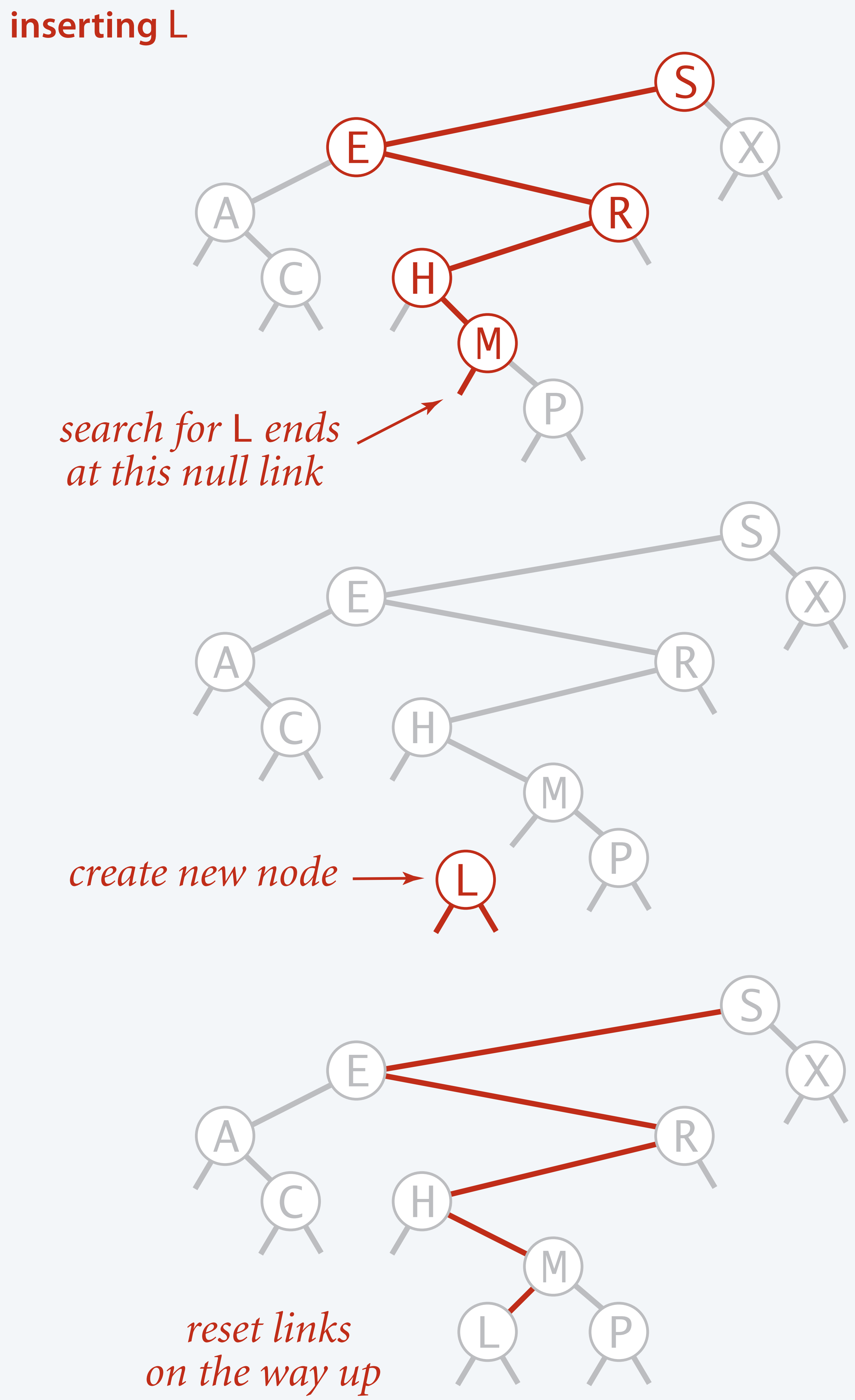
BST insert

Put. Associate value with key.

- Search for key in BST.
- Case 1: Key in BST ⇒ reset value.
- Case 2: Key not in BST ⇒ add new node.

```
public void put(Key key, Value val) {
   root = put(root, key, val);
}

private Node put(Node x, Key key, Value val) {
   if (x == null) return new Node(key, val);
   int cmp = key.compareTo(x.key);
   if      (cmp < 0) x.left  = put(x.left,  key, val);
   else if (cmp > 0) x.right = put(x.right, key, val);
   else x.val = val;

   return x;
}
```

⚠ **Warning: concise but tricky code; read carefully!**

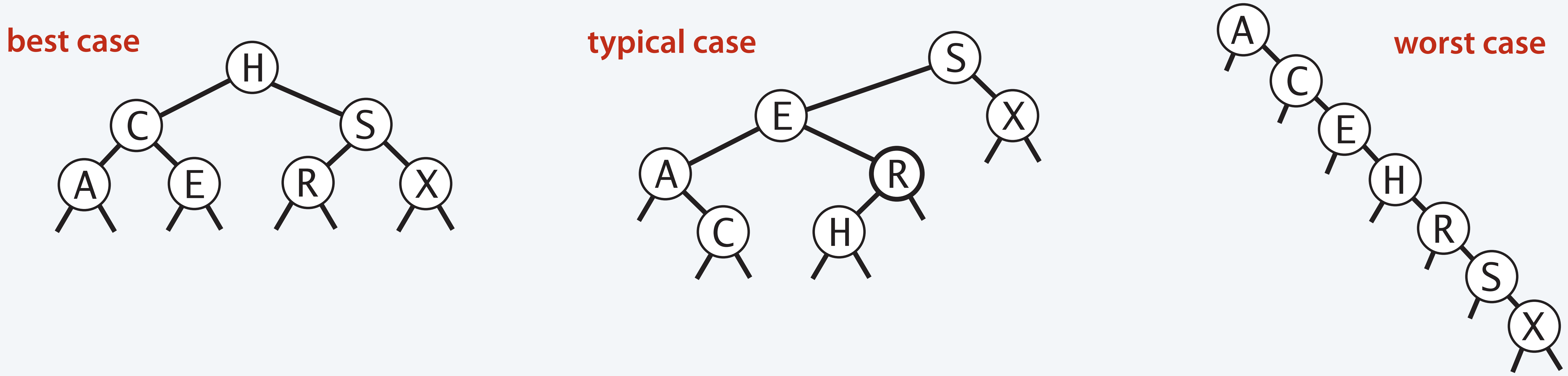


insertion into a BST

Cost. Number of compares = 1 + depth of deepest node reached.

Tree shape

- Many BSTs correspond to same set of keys.
- Number of compares for search/insert = 1 + depth of deepest node reached.

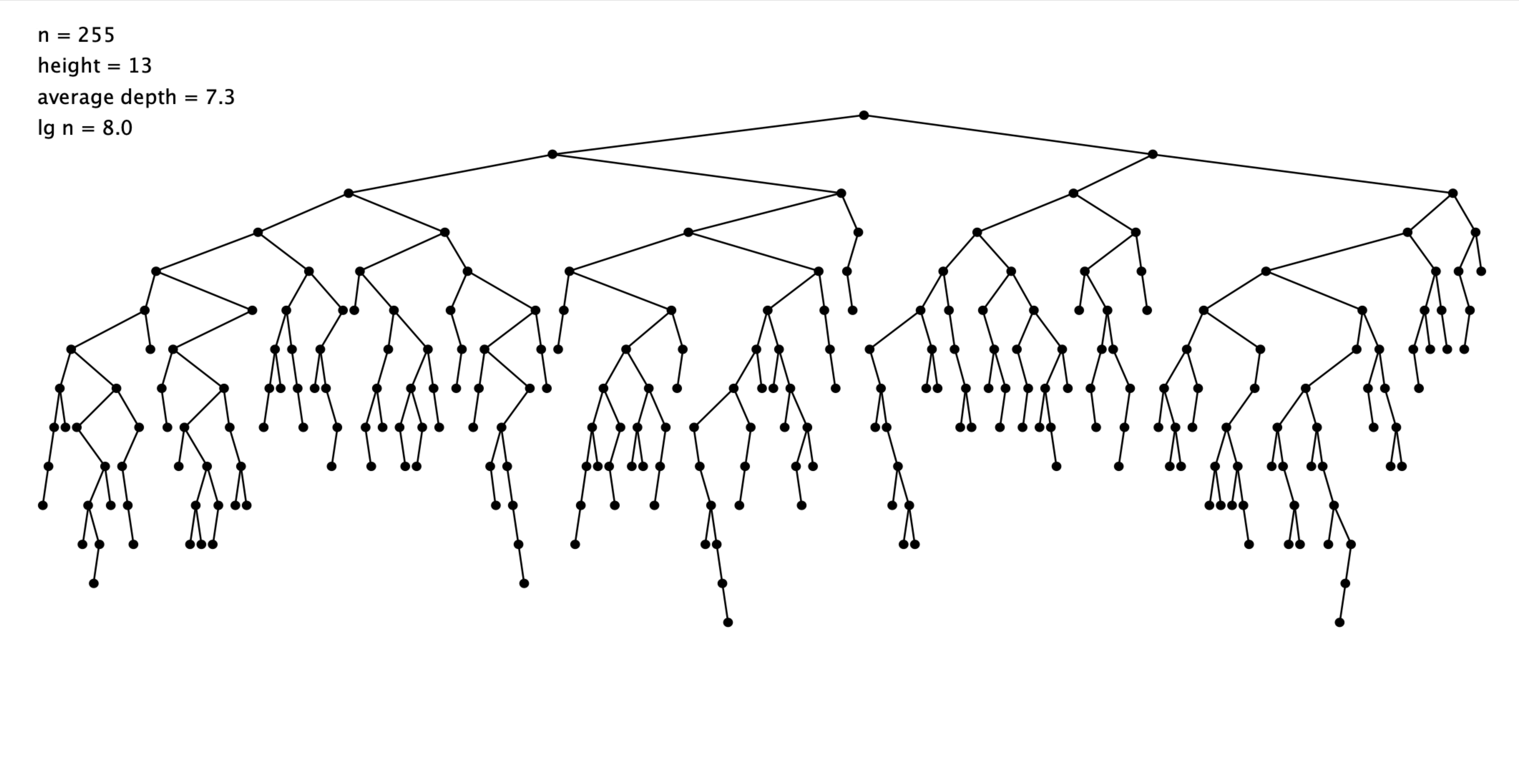


height between $\log_2 n$ *and* $n - 1$

Bottom line. Tree shape depends on order of insertion.

BST insertion: random order visualization

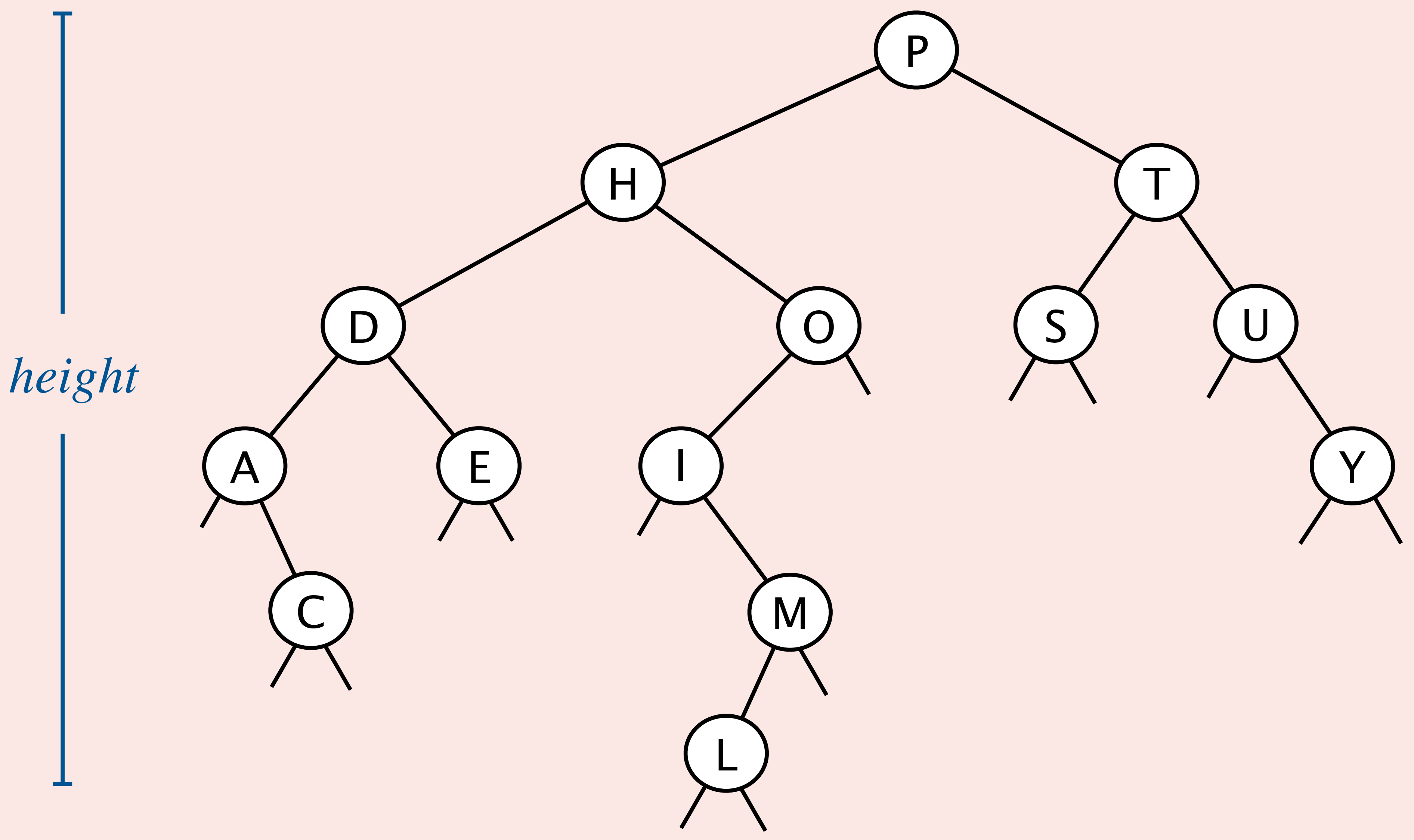
Ex. Insert 255 keys in random order.



Suppose that you insert n distinct keys in uniformly random order into a BST.

What is the expected height of the resulting BST?

- **A.** $\sim \log_2 n$
- **B.** $\sim 2 \ln n$
- **C.** $\sim 4.31107 \ln n$
- **D.** $\sim \frac{1}{2} n$
- **E.** $\sim n$



ST implementations: performance summary

implementation	worst case		typical case		operations on keys
	search	insert	search hit	insert	
sequential search (unordered list)	n	n	n	n	`equals()`
binary search (ordered array)	$\log n$	n	$\log n$	n	`compareTo()`
BST	n	n	$\log n$	$\log n$	`compareTo()`

Why not shuffle to ensure a (probabilistic) guarantee of $\Theta(\log n)$ *time à la quicksort ?*

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3.2 Binary Search Trees

- BSTs
- *iteration*
- ordered operations

Binary search trees: poll 3

In which order does `traverse(root)` print the keys in the BST?

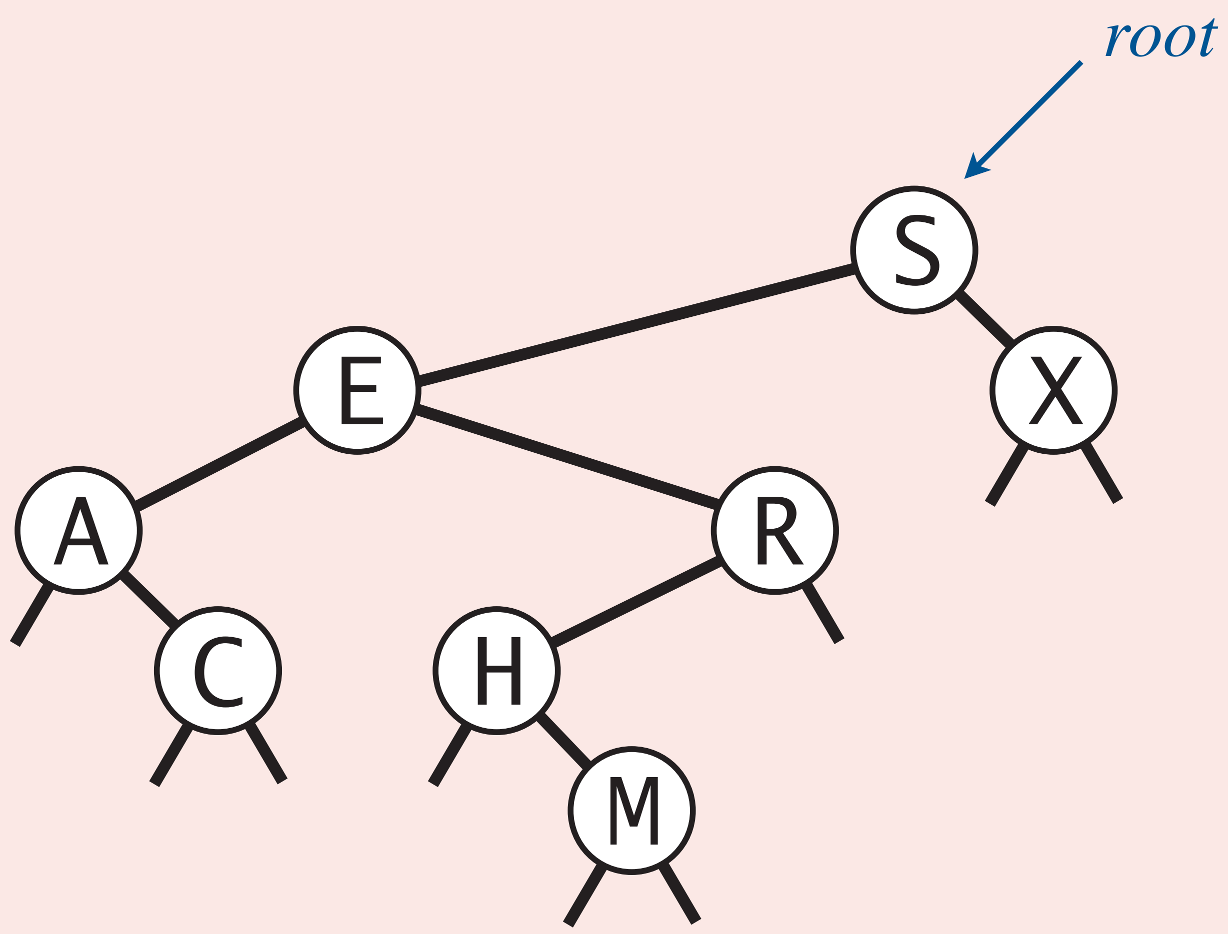
```
private void traverse(Node x) {
   if (x == null) return;
   traverse(x.left);
   StdOut.println(x.key);
   traverse(x.right);
}
```

A. A C E H M R S X

B. S E A C R H M X

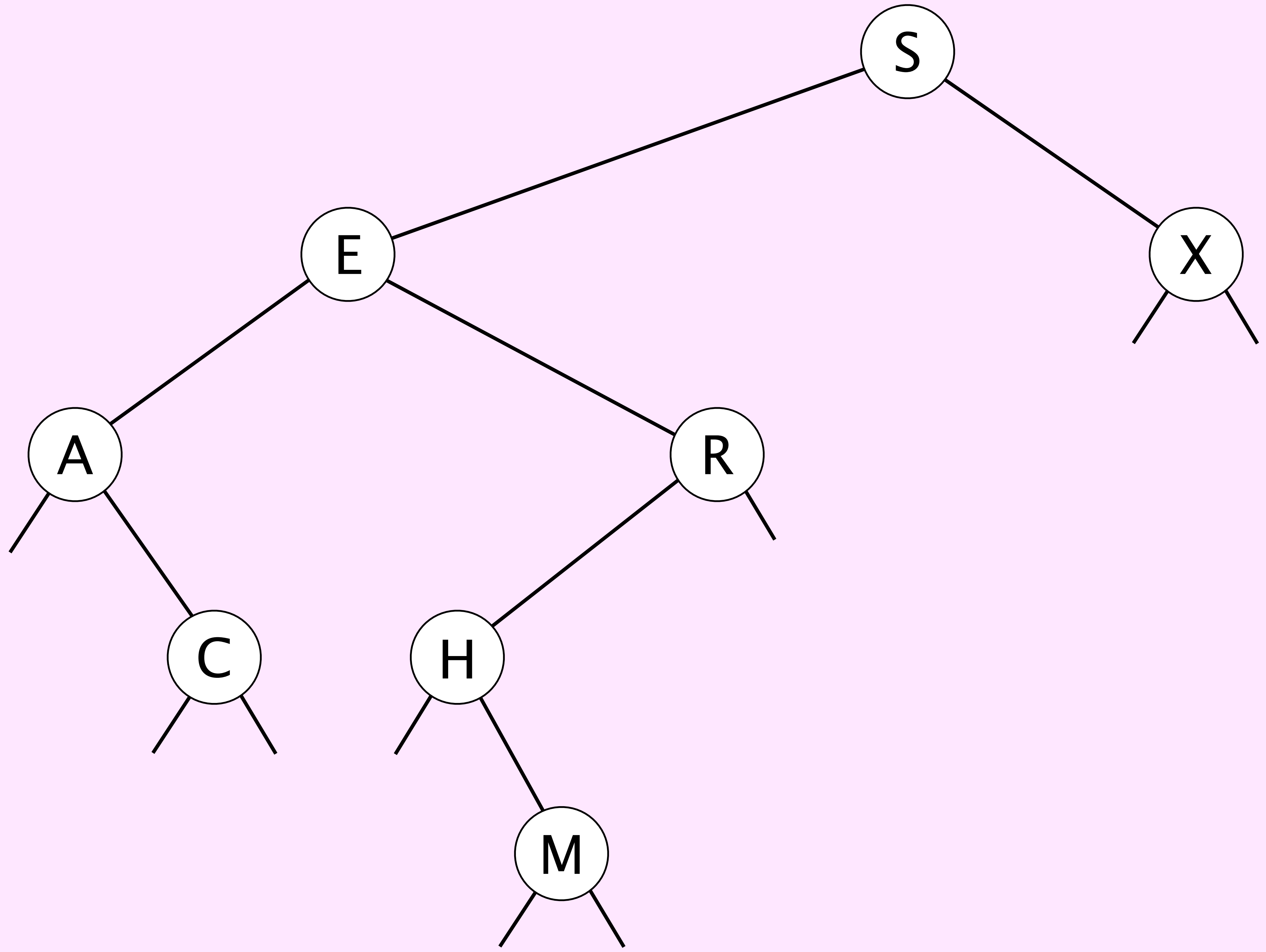
C. C A M H R E X S

D. S E X A R C H M



Inorder traversal

```
inorder(S)
   inorder(E)
      inorder(A)
         print A
         inorder(C)
            print C
            done C
         done A
      print E
      inorder(R)
         inorder(H)
            print H
            inorder(M)
               print M
               done M
            done H
         print R
         done R
      done E
   print S
   inorder(X)
      print X
      done X
   done S
```



output: A C E H M R S X

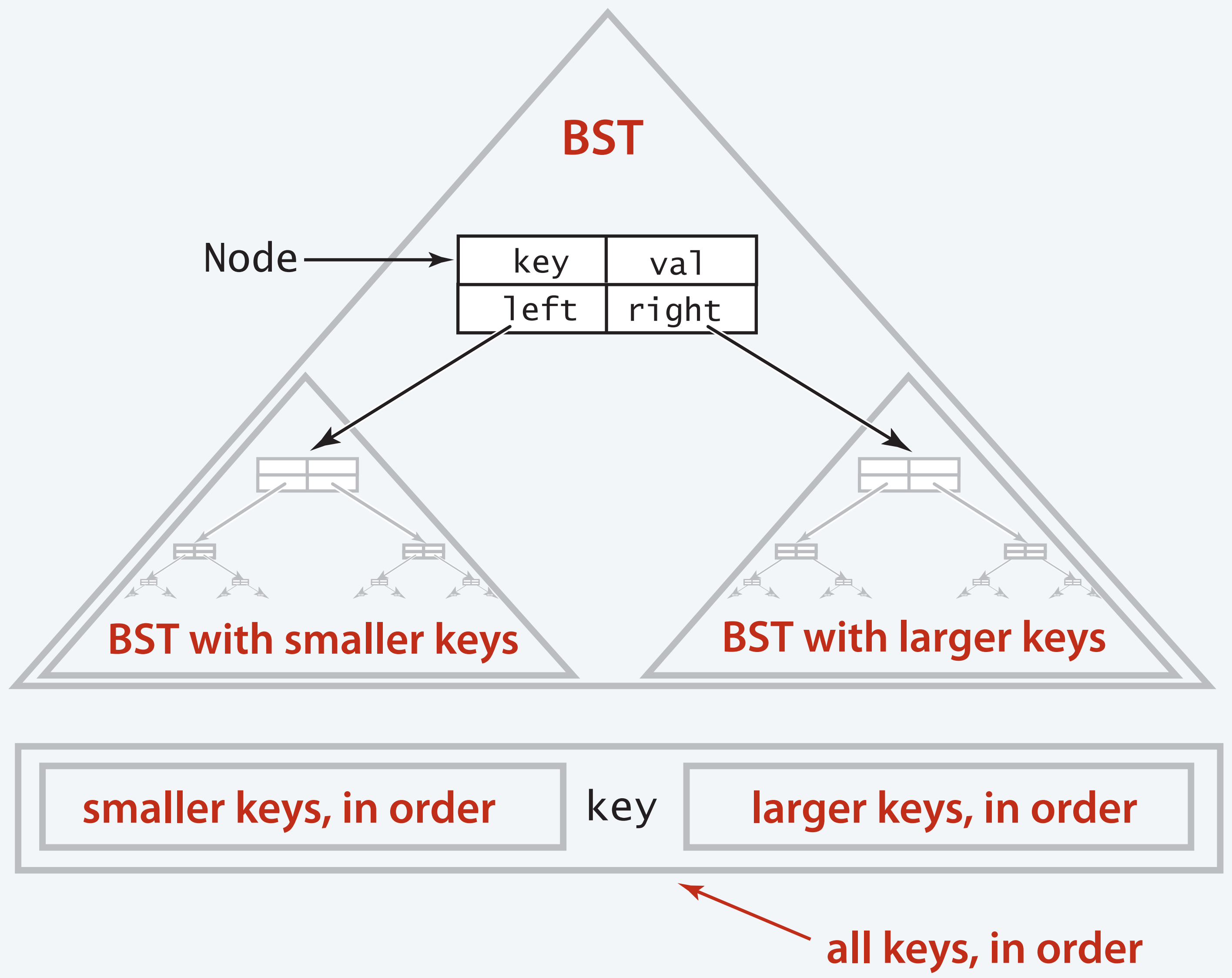
Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

add items to a collection that is `Iterable` *and return that collection*

```
public Iterable<Key> keys() {
   Queue<Key> queue = new Queue<Key>();
   inorder(root, queue);
   return queue;
}
```

```
private void inorder(Node x, Queue<Key> queue) {
   if (x == null) return;
   inorder(x.left, queue);
   queue.enqueue(x.key);
   inorder(x.right, queue);
}
```



Property. Inorder traversal of a BST yields keys in ascending order.

Inorder traversal: running time

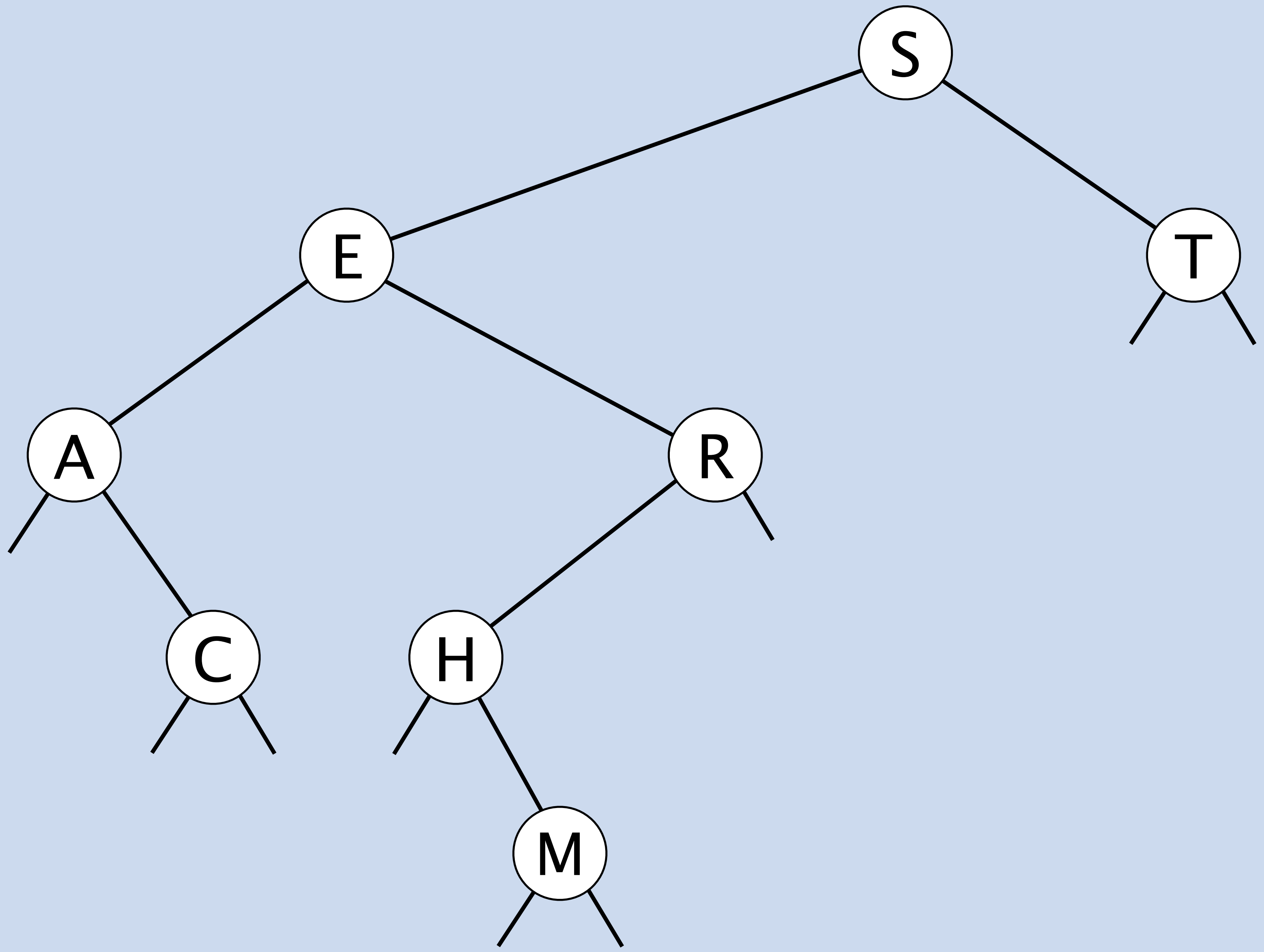
Property. Inorder traversal of a binary tree with n nodes takes $\Theta(n)$ time.

Pf. $\Theta(1)$ time per node in BST.

Level-order traversal

Level-order traversal of a binary tree.

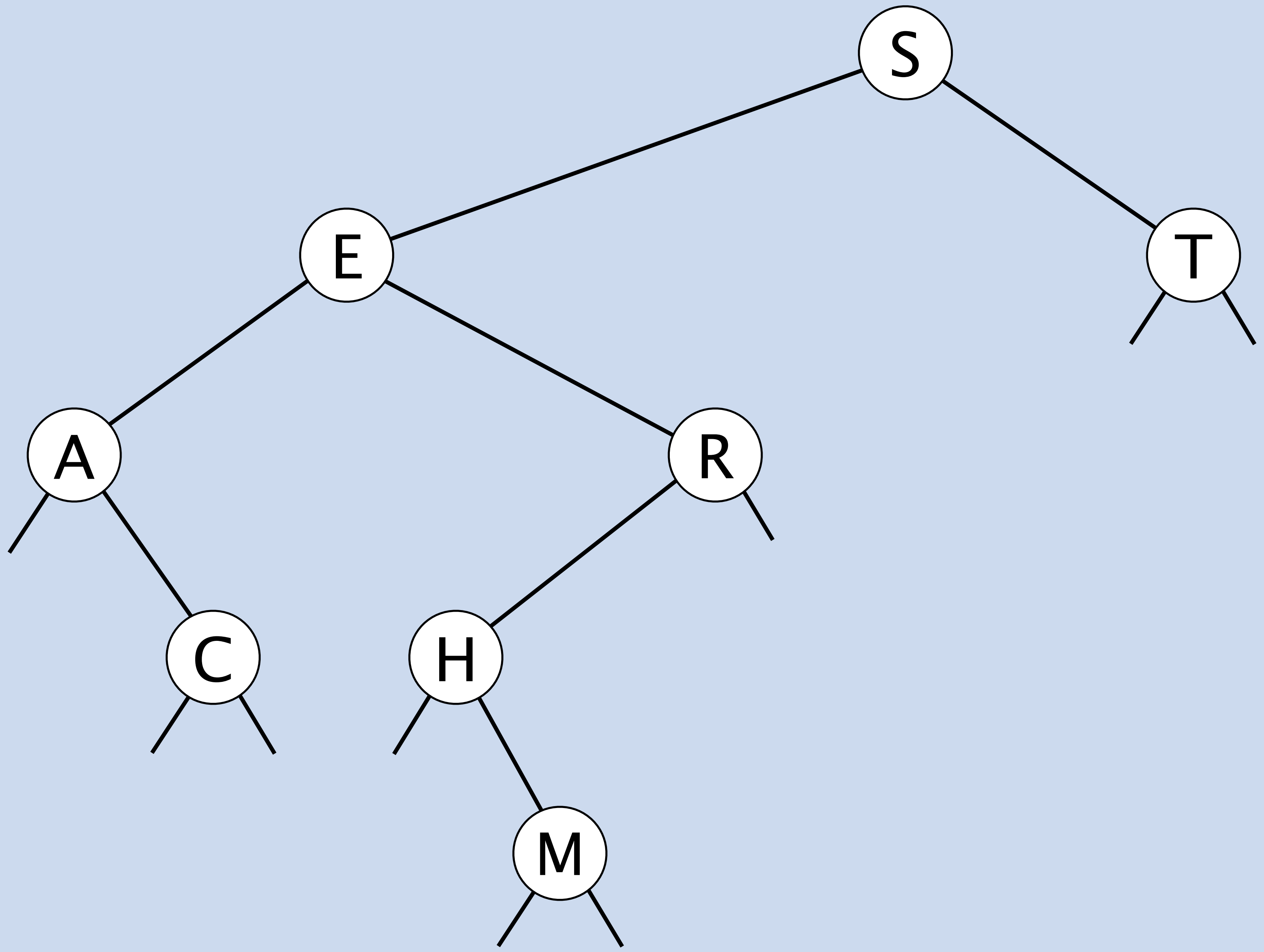
- Process root.
- Process children of root, from left to right.
- Process grandchildren of root, from left to right.
- …



level-order traversal: S E T A R C H M

Level-order traversal

Q1. How to compute level-order traversal of a binary tree in $\Theta(n)$ time?



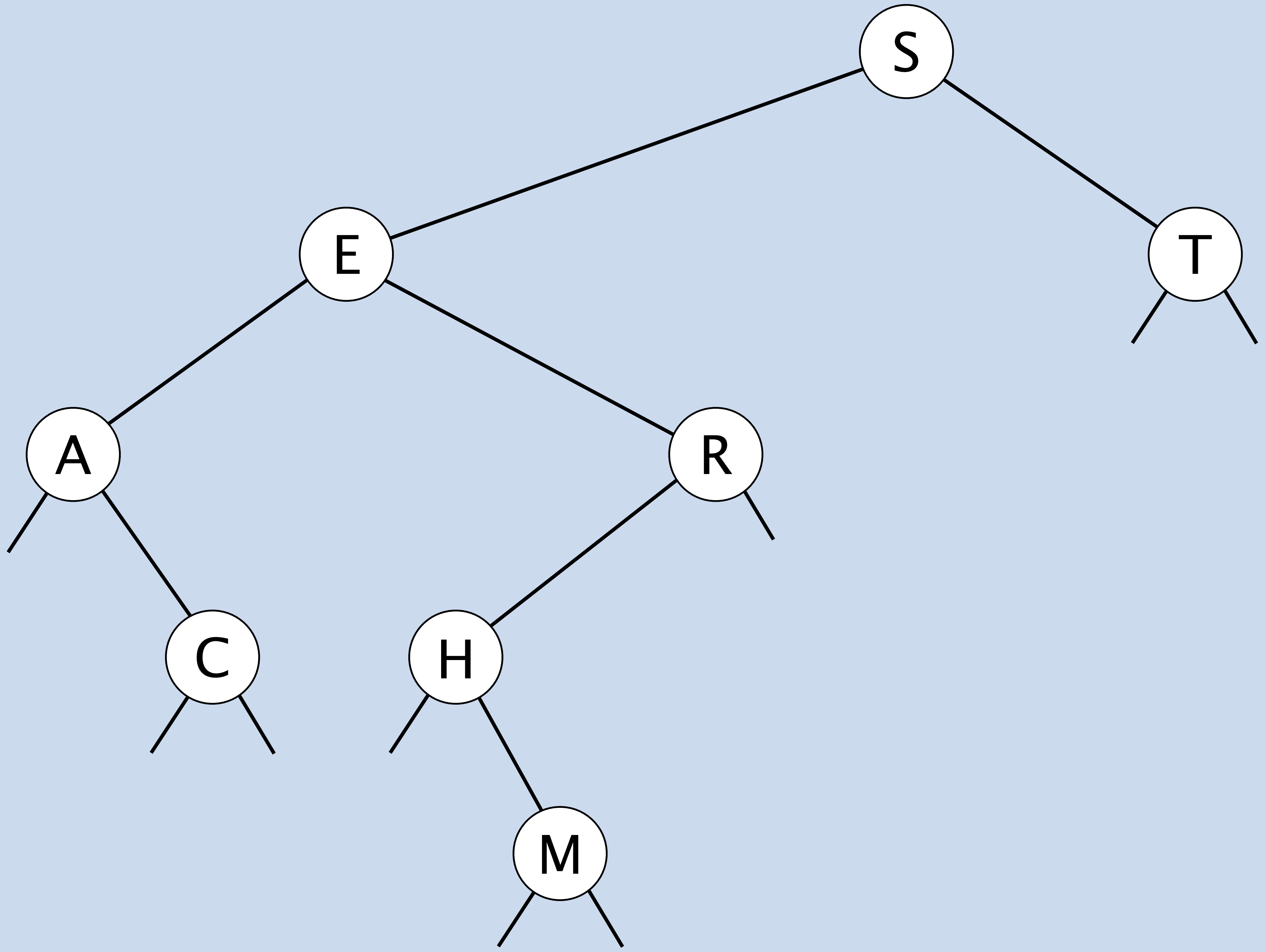
level-order traversal: S E T A R C H M

Level-order traversal

Q2. Given the level-order traversal of a BST, how to (uniquely) reconstruct?

needed for PrairieLearn quizzes

Ex. S E T A R C H M



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- iteration
- ordered operations

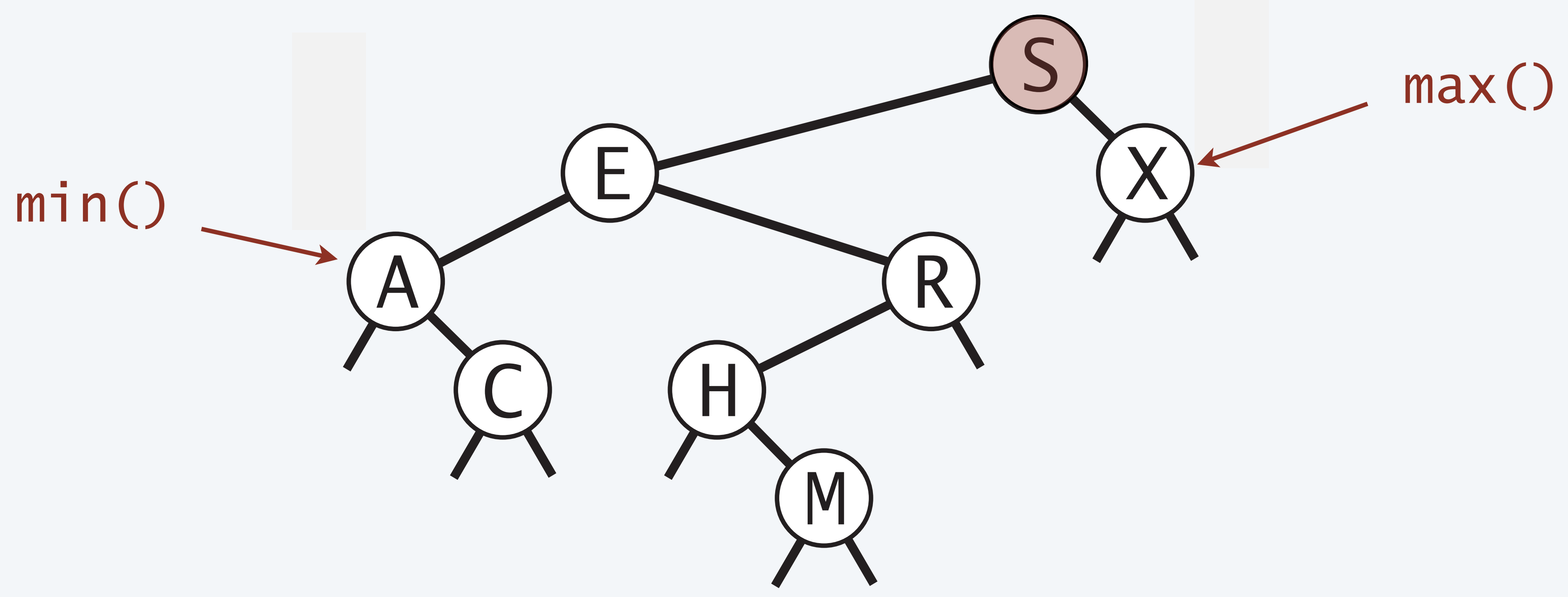
Minimum and maximum

Minimum. Smallest key in BST.

Maximum. Largest key in BST.

Q. How to find the min / max?

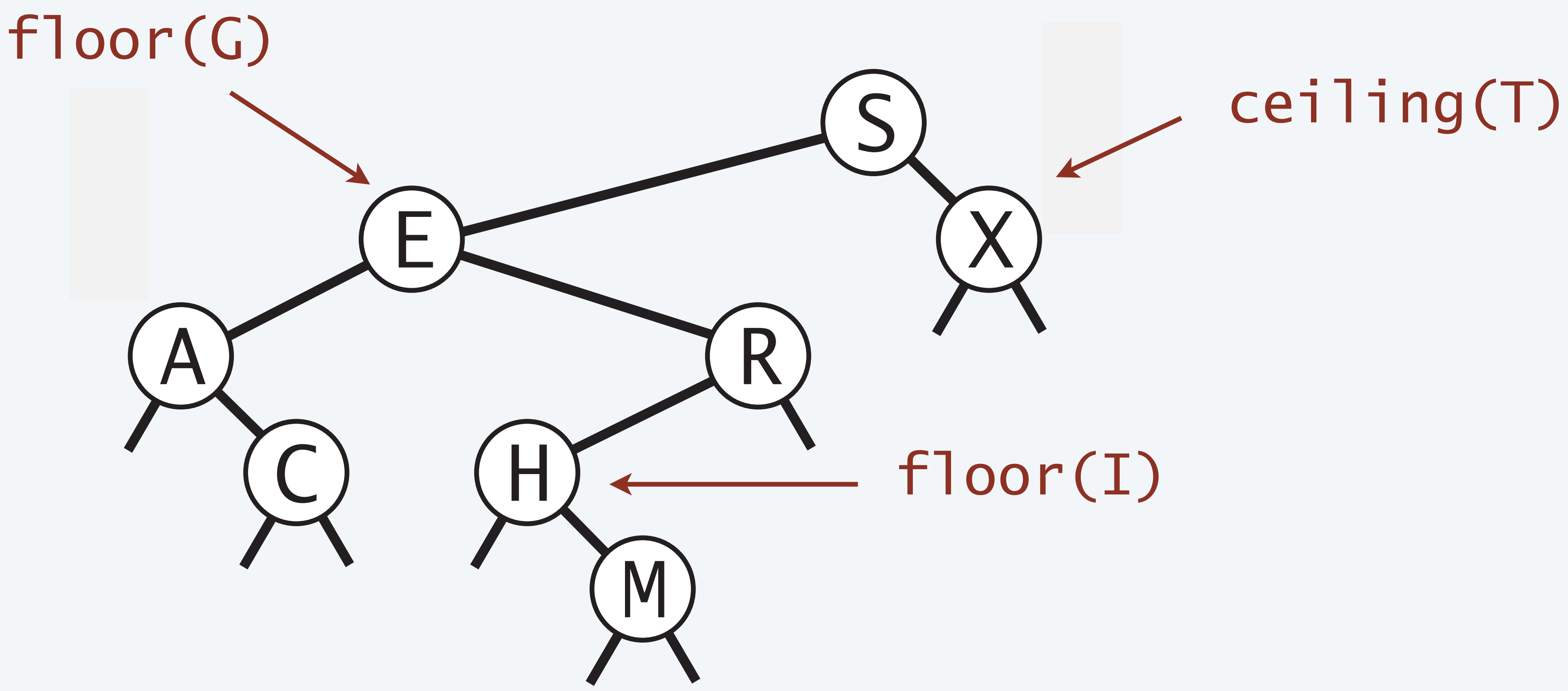
A. Go down left / right spine. ← *running time proportional to depth of node in BST (but 0 compares)*



Floor and ceiling

Floor. Largest key in BST ≤ query key.

Ceiling. Smallest key in BST ≥ query key.



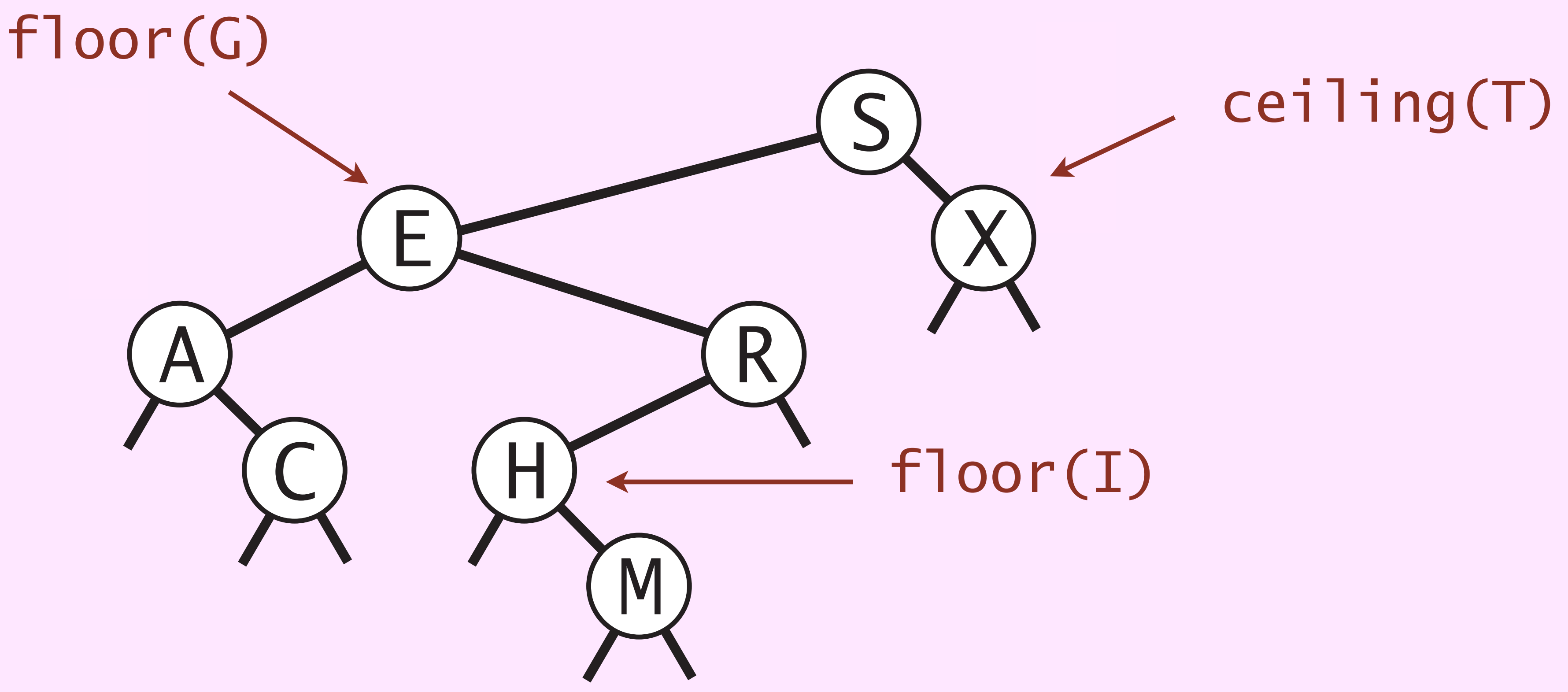
Computing the floor

Floor. Largest key in BST ≤ query key.

Ceiling. Smallest key in BST ≥ query key.

Key idea.

- To compute `floor(key)` or `ceiling(key)`, search for `key`.
- Both `floor(key)` and `ceiling(key)` are on search path.
- Moreover, as you go down search path, any candidates get better and better.



Computing the floor: Java implementation

Invariant 1. The floor is either champ or in subtree rooted at x.

Invariant 2. Node x is in the right subtree of node containing champ. ← *assuming* champ *is not null*

```
public Key floor(Key key) {
  return floor(root, key, null);
}

private Key floor(Node x, Key key, Key champ) {
   if (x == null) return champ;
   int cmp = key.compareTo(x.key);
   if      (cmp < 0) return floor(x.left, key, champ);
   else if (cmp > 0) return floor(x.right, key, x.key);
   else              return x.key;
}
```

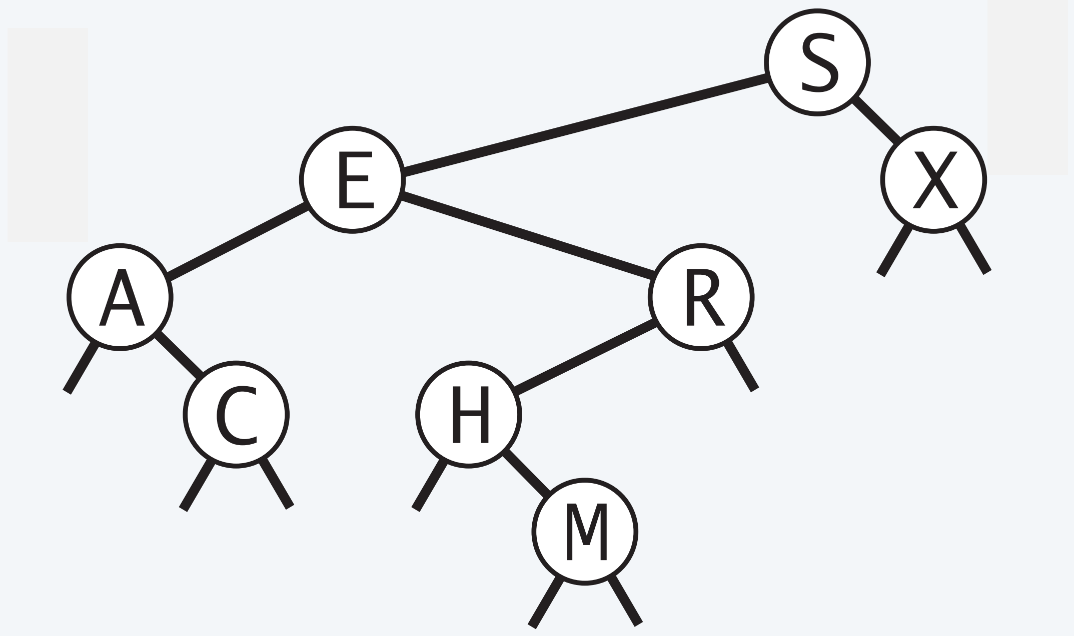
champ must be floor (annotates `return champ`)

key in node x is too large (floor can't be in right subtree of x) (annotates `floor(x.left, key, champ)`)

key is in BST (annotates `return x.key`)

key in node x is a candidate for floor (floor can't be in left subtree of x) (annotates `floor(x.right, ...)`)

key in node x is better candidate than champ (because x is in the right subtree of champ) (annotates `x.key` argument)



BST: ordered symbol table operations summary

	sequential search	binary search	BST
search	$\Theta(n)$	$\Theta(\log n)$	$\Theta(h)$
insert / delete	$\Theta(n)$	$\Theta(n)$	$\Theta(h)$
min / max	$\Theta(n)$	$\Theta(1)$	$\Theta(h)$
floor / ceiling	$\Theta(n)$	$\Theta(\log n)$	$\Theta(h)$
rank	$\Theta(n)$	$\Theta(\log n)$	$\Theta(h)$
select	$\Theta(n)$	$\Theta(1)$	$\Theta(h)$

h = height of BST

worst-case running time of ordered symbol table operations

ST implementations: summary

implementation	worst case		ordered ops?	key interface
	search	insert		
sequential search (unordered list)	n	n		`equals()`
binary search (sorted array)	$\log n$	n	✔	`compareTo()`
BST	n	n	✔	`compareTo()`
red-black BST	$\log n$	$\log n$	✔	`compareTo()`

next lecture: BST whose height is guarantee to be $\Theta(\log n)$

Credits

image	source	license
Inorder Traversal in a BST	Silicon Valley S4E5	
Binary Tree	Daniel Stori	

A final thought