Long-Range Order in a Two-Dimensional Dynamical *XY* Model: How Birds Fly Together

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We propose a nonequilibrium continuum dynamical model for the collective motion of large groups of biological organisms (e.g., flocks of birds, slime molds, etc.) Our model becomes highly nontrivial, and different from the equilibrium model, for $d < d_c = 4$; nonetheless, we are able to determine its scaling exponents *exactly* in $d = 2$ and show that, unlike equilibrium systems, our model exhibits a broken continuous symmetry even in $d = 2$. Our model describes a large universality class of microscopic rules, including those recently simulated by Vicsek *et al.*

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The dynamics of "flocking" behavior among living things, such as birds, slime molds, and bacteria, has long been a mystery. Recently, a number of simple numerical models that exhibit such behavior have been studied [1,2]. For example, Ref. [2] considers a synchronous, discrete time step rule in which an individual "bird" in a group of "birds" determines its next direction of motion on each time step by averaging the direction of its neighbors in a certain area, and then adding some zero mean noise, while keeping the magnitude of its velocity constant. Their simulations in two dimensions find a transition between an ordered phase in which the mean velocity of the flock $\langle\vec{v}\rangle \neq 0$ and a disordered phase with $\langle\vec{v}\rangle = 0$ as the strength of the noise is increased.

The above two-dimensional model is very similar to the 2D *XY* model [3,4] because the velocity of the "bird," like the local spin of the classical *XY* model, also has fixed length and continuous rotational symmetry. Indeed, it is easy to see that, in the limit that the magnitude of the velocity goes to zero, on each time step the "birds" are just picking a new direction, but never actually move, the model reduces *precisely* to the Monte Carlo dynamics of a two-dimensional *XY* model, with the (small) bird velocity playing the role of the *XY* spin. Since the 2D *XY* model does *not* exhibit a long-range ordered phase at temperatures $T > 0$ (due to spin wave fluctuations), the long-range ordered state observed in Ref. [2] seems very surprising. Indeed, in light of the Mermin-Wagner theorem [5] for equilibrium systems, its existence must depend on fundamentally dynamical, nonequilibrium aspects of the model. In this paper, we show, using a continuum dynamical equation which describes a large universality class of related dynamical models, that this is indeed the case. In particular, we explicitly demonstrate the following: (1) that our model differs from the equilibrium system for spatial dimensions $d < 4$, (2) we can calculate the scaling exponents of this model *exactly* for $d = 2$, and (3) the model does, indeed, have a stable spontaneous symmetry broken state even in two dimensions.

Our starting point is the continuum equations of motion (EOM) [6]:

$$\partial_t\vec{v} + (\vec{v}\cdot\nabla)\vec{v} = \alpha\vec{v} - \beta|\vec{v}|^2\vec{v} - \nabla P + D_L\nabla(\nabla\cdot\vec{v}) + D_1\nabla^2\vec{v} + D_2(\vec{v}\cdot\nabla)^2\vec{v} + \vec{f}\,, \tag{1}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\vec{v}\rho) = 0\,, \tag{2}$$

where β, D_1, D_2, and D_L are all positive, and $\alpha < 0$ in the disordered phase and $\alpha > 0$ in the ordered state. The left hand side of Eq. (1) is just the usual convective derivative of the coarse-grained velocity field $\vec{v}$. The α and β terms simply make the local $\vec{v}$ have a nonzero magnitude ($= \sqrt{\alpha/\beta}$) in the ordered phase. $D_{L,1,2}$ are diffusion constants. The Gaussian random noise $\vec{f}$ has correlations:

$$\langle f_i(\vec{r},t)f_j(\vec{r}\,',t')\rangle = \Delta\delta_{ij}\delta^d(\vec{r}-\vec{r}\,')\delta(t-t')\,,$$

where Δ is a constant and i and j denote Cartesian components. Finally, the pressure

$$P = P(\rho) = \sum_{n=1}^{\infty}\sigma_n(\rho-\rho_0)^n,$$

where ρ_0 is the mean of the local number density and $\rho(\vec{r})$ and σ_n are coefficients in the pressure expansion. The final equation (2) reflects conservation of birds.

The essential difference between our model and the equilibrium *XY* model is the existence of the convective term in our model, which makes the dynamics nonpotential and further stabilizes the ordered phase. A heuristic argument for the stabilizing effect of the convective term can be given if we consider our model in Lagrangian coordinates. In those coordinates, the convective term drops out and the interaction between the velocity field is local at each instance. However, at different times, the "neighbors" of one particular bird will be different depending on the velocity field itself. Therefore, two originally distant birds can interact with each other at some later time. It is exactly this time dependent variable ranged interaction which stabilizes the ordered phase.

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To treat the problem analytically, it is more convenient to use the Eulerian coordinates as in Eqs. (1) and (2). In the rest of our paper, we concentrate on studying the symmetry broken phase, where $\alpha > 0$. We can write the velocity field as $\vec{v} = v_0\hat{x}_\parallel + \delta\vec{v}$, where $v_0\hat{x}_\parallel = \langle\vec{v}\rangle$ is the spontaneous average value of $\vec{v}$. We will ignore fluctuations in the magnitude $|\vec{v}|$ from its optimal value of $\sqrt{\alpha/\beta}$, since they decay in a finite time (of order $\frac{1}{\alpha}$). Choosing our units of velocity so that $\sqrt{\alpha/\beta} = 1$, we can now write the velocity as $\vec{v} = (\vec{v}_\perp, \sqrt{1 - |\vec{v}|^2}) \sim (\vec{v}_\perp, 1 - \frac{1}{2}|\vec{v}_\perp|^2)$, provided $|\vec{v}_\perp|^2 \ll 1$.

Shifting to a comoving coordinate frame moving with velocity $\vec{v}_0\hat{x}_\parallel$, $\vec{v} = (\vec{v}_\perp, -\frac{1}{2}|\vec{v}_\perp|^2)$, and the convective term becomes $(\vec{v}_\perp \cdot \nabla_\perp)\vec{v}_\perp - \frac{1}{2}|\vec{v}_\perp|^2\partial_\parallel\vec{v}_\perp$. We will neglect the second term; this will be justified *a posteriori*. The equation of motion then becomes

$$\partial_t\vec{v}_\perp + \lambda(\vec{v}_\perp \cdot \vec{\nabla}_\perp)\vec{v}_\perp = -\nabla_\perp P + D_L\nabla_L(\nabla_\perp \cdot \vec{v}_\perp) + D_1\nabla_\perp^2\vec{v}_\perp + D_\parallel\partial_\parallel^2\vec{v}_\perp + \vec{f}_\perp, \tag{3}$$

$$\frac{\partial\delta\rho}{\partial t} + \rho_0\nabla_\perp \cdot \vec{v}_\perp + \lambda\nabla \cdot (\vec{v}\delta\rho) = 0, \tag{4}$$

with $D_\parallel = D_1 + D_2$ and $\delta\rho = \rho - \rho_0$. The bookkeeping coefficient $\lambda = 1$ in the physical case.

We first study the linearized EOM. We rescale lengths, time, and the fields $\vec{v}_\perp$ and $\delta\rho$ according to

$$\vec{x}_\perp \to b\vec{x}_\perp, \quad x_\parallel \to b^\zeta x_\parallel, \quad t \to b^z t, \quad \vec{v}_\perp \to b^\chi\vec{v}_\perp, \quad \delta\rho \to b^{\chi_\rho}\delta\rho. \tag{5}$$

We choose the scaling exponents to keep the diffusion constants $D_1, D_\parallel, D_\perp = D_1 + D_L$ and the strength Δ of the noise fixed. The reason for choosing to keep these particular parameters fixed rather than, e.g., σ_1, is that these four parameters completely determine the size of the equal time fluctuations in the linearized theory, as can be seen by solving that theory in Fourier space:

$$\langle v_i^\perp(\vec{q},t)v_j^\perp(-\vec{q},t)\rangle = \frac{\Delta}{2}\left(\frac{q_i^\perp q_j^\perp}{(D_\perp q_\perp^2 + D_\parallel q_\parallel^2)q_\perp^2} + \frac{q_\perp^2\delta_{ij}^\perp - q_i^\perp q_j^\perp}{(D_1 q_\perp^2 + D_\parallel q_\parallel^2)q_\perp^2}\right).$$

The exponents for the linear theory can be determined very easily: $z = 2$, $\zeta = 1$, $\chi = 1 - d/2$, and $\chi_\rho = \chi$, because the density fluctuations of $\delta\rho$ are comparable to those of $\vec{v}_\perp$. Therefore the linearized theory implies that $\vec{v}_\perp$ fluctuations grow without bound (like L^χ) as $L \to \infty$ for $d \leq 2$, where the above expression for χ becomes positive. This implies the loss of long-range order in $d \leq 2$.

Making the rescalings as described above, the other parameters in the model scale as $\lambda \sim b^{\gamma_\lambda}\lambda$, $\sigma_n \sim b^{\gamma_n}\sigma_n$ with $\gamma_\lambda = \chi + 1 = 2 - d/2$ and $\gamma_n = z - \chi + n\chi = n + (1 - n)d/2$. The first of these scaling exponents to become positive with decreasing d are γ_λ and γ_2, which both do so for $d < 4$, indicating that the $\lambda(\vec{v}_\perp \cdot \vec{\nabla})\vec{v}_\perp$ and $\sigma_2\vec{\nabla}_\perp(\delta\rho^2)$ nonlinearities are both relevant perturbations for $d < 4$. So for $d < 4$, the linearized hydrodynamics will *break down.*

An $\epsilon = 4 - d$ expansion will obviously not be of much use in our problem in $d = 2$. But fortunately, because of the various symmetries in Eqs. (3) and (4), we can obtain the *exact* scaling exponents in $d = 2$. First of all, the reduced equations of motion Eqs. (3) and (4) have a "Galilean invariance" [7]: i.e., if we let $\vec{v}_\perp(\vec{r},t) \to \vec{v}_\perp(\vec{r},t) + \vec{v}_{\perp,0}$ and simultaneously boost the coordinate $\vec{x}_\perp \to \vec{x}_\perp - \lambda\vec{v}_{\perp,0}t$, Eqs. (3) and (4) remain invariant for arbitrary values of $\vec{v}_{\perp,0}$. This implies that there is no "graphical" renormalization of the nonlinear vertex $(\vec{v}_\perp \cdot \nabla_\perp)\vec{v}_\perp$; it can only renormalize by rescaling. Furthermore, in precisely two dimensions, $D_\parallel$ and Δ are also only renormalized by rescaling. To see this, note that in two dimensions $\vec{v}_\perp$ has only one component (call it v_x), which can be written as $v_x = \partial_x h$. The equations of motion (3) and (4) can then be rewritten in terms of h:

$$\partial_t h + \frac{\lambda}{2}|\nabla_\perp h|^2 = -\nabla_\perp P + D_\perp\nabla_\perp^2 h + D_\parallel\partial_\parallel^2 h + \eta, \tag{6}$$

$$\frac{\partial\delta\rho}{\partial t} + \rho_0\nabla_\perp^2 h + \lambda\nabla_\perp \cdot (\delta\rho\nabla_\perp h) = 0, \tag{7}$$

where the new effective noise $\eta = \vec{\nabla}_\perp \cdot \vec{f}/\nabla_\perp^2$ has long-ranged correlations. In Fourier space

$$\langle\eta(\vec{q},\omega)\eta(-\vec{q}\,',-\omega')\rangle = \Delta\frac{\delta^d(\vec{q} - \vec{q}\,')\delta(\omega - \omega')}{q_\perp^2}.$$

This model only corresponds to our original model (3) and (4), and hence only describes birds, in $d = 2$. However, we will analyze this model (6) and (7) in arbitrary spatial dimensions d, with the goal of understanding it in the physical case $d = 2$.

It is easy to see that $D_\parallel$ and Δ cannot be renormalized in this model. Δ cannot be renormalized because it is the coefficient of a nonanalytic, long-ranged noise-noise correlation function. The nonlinear couplings λ and σ_n in (6) and (7), being analytic (i.e., local in space and time), can therefore *not* generate such nonanalytic correlations.

The diffusion constant $D_\parallel$ likewise cannot be renormalized, because any such renormalization must clearly involve at least one λ with at least one external h leg, and hence at least one power of $q_\perp$. This can (and does) renormalize $D_\perp$, but cannot renormalize $D_\parallel$.

Implementing the dynamical renormalization group [8,9] we find, quite generally

$$\frac{dD_{\|}}{dl} = (z - 2\zeta)D_{\|},$$
$$\frac{d\Delta}{dl} = (z + 1 - d - \zeta - 2\chi)\Delta,$$
$$\frac{d\lambda}{dl} = (\chi + z - 1)\lambda,$$
$$\frac{dD_{\perp}}{dl} = [z - 2 + G_{\perp}(\{g_m\})]D_{\perp},$$
$$\frac{d\rho_0}{dl} = [z - 1 + G_{\rho}(\{g_m\})\Gamma]\rho_0,$$
$$\frac{d\sigma_1}{dl} = [z - 1 + G_1(\{g_m\})\Gamma]\sigma_1,$$
$$\frac{d\sigma_n}{dl} = \left[z + (n-1)\chi - 1 + \frac{G_n(\{g_m\})}{g_n}\right]\sigma_n, \quad (8)$$

with the parameter $\Gamma = D_{\perp}/\sqrt{\sigma_1\rho_0}$, and the effective nonlinear coupling constants $g_1 = \lambda\Delta^{1/2}/D_{\perp}^{5/4}D_{\|}^{1/4}$ and $g_{n\geq 2} = \sigma_n\Delta^{(n-1)/2}\rho_0^{n/2}/D_{\perp}^{(n+3)/4}D_{\|}^{(n-1)/4}\sigma_1^{n/2}$. The $G_{\perp,\rho,n}$'s denote the nonvanishing graphical corrections to $D_{\perp}$, ρ_0, and σ_n, respectively. Note that they explicitly depend on *only* the coupling constant's g_m's, by construction. Note that the absence of graphical corrections to $D_{\|}$, Δ, and λ is *exact* to all orders in perturbation theory, as discussed earlier. Since we seek a fixed point at which Δ, λ, and D remain fixed, we get the following three exact constraints on the three exponents:

$$\chi + z = 1, \quad z = 2\zeta, \quad d + \zeta + 2\chi = z + 1, \quad (9)$$

whose solution is

$$z = \frac{2(d+1)}{5}, \quad \zeta = \frac{d+1}{5}, \quad \chi = \frac{3-2d}{5}. \quad (10)$$

If we combine the above renormalization group (RG) equations (8), we can obtain the RG flow equations for the effective coupling constants g_n, and the parameter Γ:

$$\frac{d\Gamma}{dl} = -[1 - G_{\perp}(\{g_m\})]\Gamma - \tfrac{1}{2}[G_1(\{g_m\}) + G_{\rho}(\{g_m\})]\Gamma^2, \quad (11)$$

$$\frac{dg_1}{dl} = \tfrac{1}{2}[4 - d - \tfrac{5}{2}G_{\perp}(\{g_m\})]g_1, \quad (12)$$

$$\frac{dg_{n(>1)}}{dl} = \tfrac{1}{2}[2n + (1-n)d]g_n + G_n(\{g_m\}) + \frac{n}{2}(G_{\rho} - G_1)\Gamma - \frac{n+3}{4}G_{\perp}, \quad (13)$$

from which we see that g_1 and g_2 become relevant below $d = 4$, while all $g_{n>2}$ are *irrelevant* near $d = 4$. Hence we can neglect all $g_{n>2}$'s, and, therefore, all of the $\sigma_{n>2}$'s as well, at least near $d = 4$. The parameter Γ only becomes relevant below $d = 1.5$, where $\chi > 0$, i.e., when the ordered phase disappears. We have calculated the graphical corrections $G_{\perp}$ and G_2 to one loop order near $d = 4$, and obtain $G_{\perp} = (11/192\pi^2)(g_1/2 + g_2)g_1/2$, $G_2 = 0$. We do not know whether the vanishing of G_2 to this order is the result of some symmetry of the problem that we have failed to recognize, in which case G_2 would vanish to all orders, or if it is purely coincidental. In either case, to one loop order, inserting these results for the graphical corrections into the recursion relations (16) yields a fixed *line* (actually a fixed hyperbola) $(g_1/2 + g_2)g_1 = 768\pi^2\epsilon/55$. This summarizes our picture for model (6) and (7) in $4 - \epsilon$ dimensions. What happens as we move down to two dimensions, which is the only dimension in which the model (6) and (7) actually describes birds? If G_2 vanishes to all orders in ϵ, we will still get a fixed line, as in $4 - \epsilon$ dimensions, all the way down to $d = 2$, although its position will be shifted (and it might become curved) away from that given by the one loop calculation. If G_2 does not remain zero, then the fixed line collapses to a fixed point. In either case, the scaling exponents continue to be given by Eq. (10), since those results depended only on the symmetries of the model.

So in $d = 2$, the exponents are given by Eq. (10), i.e., $z = \frac{6}{5}$, $\zeta = \frac{3}{5}$, and $\chi = -\frac{1}{5}$. These exponents can be checked experimentally (or from simulations) by measuring, e.g., the density-density correlation function, which is given, in Fourier space, by

$$\langle|\rho(\vec{q},\omega)|^2\rangle = \frac{\Delta q_{\perp}^2\rho_0^2}{(\omega^2 - c^2q_{\perp}^2)^2 + \omega^2[D_{\perp}^R(\vec{q},\omega)q_{\perp}^2 + D_{\|}q_{\|}^2]^2}, \quad (14)$$

where $c = \sqrt{\sigma_1\rho_0}$ is the speed of sound and $D_{\perp}^R$ is the renormalized diffusion constant. As a function of ω, this correlation function (like all of the correlation and response functions for this problem) has two sharp peaks at $\omega = \pm cq_{\perp}$, of width $D_{\perp}^R q_{\perp}^2 + D_{\|}q_{\|}^2$. Thus, in the frequency regime containing most of the weight of the correlation function, $D_{\perp}^R(q_{\perp}, q_{\|}, \omega)$ can be evaluated at $\omega = cq_{\perp}$. Using standard renormalization group arguments and the recursion relation (8) for $D_{\perp}$, we find that

$$D_{\perp}^R(\vec{q}_{\perp}, q_{\|}, \omega \sim cq_{\perp}; \lambda, \rho_0, \sigma_n) = q_{\perp}^{z-2}f\left(\frac{q_{\|}}{q_{\perp}^{\zeta}}\right). \quad (15)$$

Similar RG arguments yield the finite size scaling of the real-space, real-time rms fluctuations of $\vec{v}_{\perp}$:

$$\langle|\vec{v}_{\perp}(\vec{r},t)|^2\rangle = \text{const} - L_{\perp}^{2\chi}g\left(\frac{L_{\|}}{L_{\perp}^{\zeta}}\right) = \text{const} - L_{\perp}^{-2/5}g\left(\frac{L_{\|}}{L_{\perp}^{3/5}}\right), \quad (16)$$

where $L_{\perp}$ and $L_{\|}$ are the spatial dimensions of the flock perpendicular to and along the mean direction of motion, respectively, and in the last equality we have used the value of χ in $d = 2$. Since this goes to a finite constant as $L \rightarrow \infty$, we see that long-ranged order *is stable* in this model in $d = 2$, as we claimed in the Introduction.

Now that we have obtained all the exponents, we need to return to our original model (1) and (2) and verify *a posteriori* all of our assumptions. In particular, we must show that it was valid to neglect $|\vec{v}_\perp|^2 \partial_\| \vec{v}_\perp$, which, under the rescalings Eq. (3) scales like $\sim b^{2\chi+z-\zeta} \equiv b^\delta$. Using the linearized results for χ, z, and ζ, we get $\delta = 2\chi + z - \zeta = 3 - d$, which is clearly less than zero near $d = 4$. Does it remain <0 down to $d = 2$? Experience with, e.g., the $4 - \epsilon$ expansion for the ϕ^4 theory of critical phenomena suggests that it does. In that problem, a ϕ^6 perturbation also has a *linearized* RG eigenvalue $3 - d$. This term nonetheless appears to remain irrelevant all the way down to $d = 2$, judging by the success of extrapolations [10] of $4 - \epsilon$ results for the Ising model down to $d = 2$. The apparent contradiction between this result and the eigenvalue $3 - d$, which, of course, becomes positive for $d < 3$, is that graphical corrections of $O(\epsilon)$ to this result occur, and keep the eigenvalue negative down to $d = 2$. It seems just as safe to assume that this happens here as in ϕ^4 theory, and so we strongly suspect that it does, and that our results for the exponents do hold *exactly* in $d = 2$.

Even in the wildly unlikely event that the cubic vertex *does* become relevant above $d = 2$, however, we can still show that our model has long-ranged order in $d = 2$. If the cubic vertex does become relevant, we can no longer obtain the exact scaling exponents in $d = 2$, because both λ and $D_\|$ are now renormalized.

However, not all the scaling relations are lost. The random force is still renormalized, since even the contributions from the new vertex $|\vec{v}_\perp|^2 \partial_\| \vec{v}_\perp$ are proportional to $q_\|^2$, which still vanishes as $|\vec{q}| \rightarrow 0$. Furthermore, there is a new scaling relation coming from the rotational invariance, i.e., since the direction in which we choose to break the symmetry of $\vec{v}$ was arbitrary, we must, even after renormalization, be able to resum the nonlinear terms $(\vec{v}_\perp \cdot \nabla)\vec{v}_\perp$ and $|\vec{v}_\perp|^2 \partial_\| \vec{v}_\perp$ vertices into the form $(\vec{v} \cdot \nabla)\vec{v}$. This requires that the graphical corrections to $(\vec{v}_\perp \cdot \nabla)\vec{v}_\perp$ and $|\vec{v}_\perp|^2 \partial_\| \vec{v}_\perp$ be the same. To find a fixed point, therefore, we must have their rescalings to be the same as well. This leads to a new scaling relation $2\chi - 1 = 3\chi - \zeta$, or $\chi = \zeta - 1$. Taken together with the last exponents relation in (9), this leads to the following scaling relation between z and χ: $\chi = (z - d)/3$. Now we expect on physical grounds that $z < 2$ for all dimensions $d < 4$, since physically, the motion of the birds enhances the mixing, and we know the corrections due to this effect diverge below $d = 4$. Hence we expect hyperdiffusive behavior, which implies $z < 2$. Then the scaling relation $\chi = (z - d)/3$ implies $\chi < 0$, i.e., true long-range order in $d = 2$.

Numerical simulations [2] indeed find a long-range ordered state in the low "temperature" regime, in agreement with our predictions above. A detailed study of the correlation functions to test our predictions for the scaling exponents [e.g., measurements of the ρ-ρ correlation function in Eqs. (14)] would clearly be of great interest.

Considerable work remains to be done on this model. The properties of the low temperature phase of our original model (1) and (2) in $d > 2$ remain to be determined. Since the symmetries which prevent the renormalization of the noise strength Δ and the diffusion constant $D_\|$ are lost in $d > 2$, it is no longer possible to obtain exact exponents. However, an ϵ expansion on the full model (1) and (2) should give quite accurate exponents in $d = 3$.

There is also the question of the transition from the ordered to the disordered state. Without the convective vertex, our model (1) and (2) is just model A dynamics for a ϕ^4 theory, as studied by Halperin, Hohenberg, and Ma [11]. However, we can show that, as in the low temperature phase, the convective vertex becomes relevant at the transition in $d = 4$ as well. An ϵ expansion study of this problem is also currently underway. We will include these subjects and the detailed account of this Letter in a future publication [8].

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