Poly-HO
(Polymorphic, Higher-Order Programming)

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A Few More Thoughts on Types & Lists

Last Time: Java Pair Rant

Java has a paucity of types

- There is no type to describe just the pairs
- There is no type to describe just the triples
- There is no type to describe the pairs of pairs
- There is no type …

OCaml has many more types

- use option when things may be null
- do not use option when things are not null
- OCaml types describe data structures more precisely
  - programmers have fewer cases to worry about
  - entire classes of errors just go away
  - type checking and pattern analysis help prevent programmers from ever forgetting about a case

Summary of Java Pair Rant

Java has a paucity of types

- There is no type to describe j… t the pairs
- There is n… type to describe …
- There is no … describ…
- There is no t…

OCam…

- use …
- d…
  - …
  - type ch… d p… analys… help prevent programmers from ever fo… g about a case

SCORE: OCAML 1, JAVA 0

C, C++ Rant

Java has a paucity of types

- but at least when you forget something,

it ***throws an exception*** instead of ***silently going off the trolley***!

If you forget to check for null pointer in a C program,

- no type-check error at compile time
- no exception at run time
- it might crash right away (that would be best), or
- it might permit a buffer-overrun (or similar) vulnerability
- so the hackers pwn you!

Summary of C, C++ rant

Java has a paucity of types

– but at least when you forg omethi

it ***throws a ntion*** ins g off trolley!

If you

– no ty

SCORE:

OCAML 1, JAVA 0, C -1

– it n), or

– it similar ulnerability

– so the hac n y

MORE THOUGHTS ON LISTS

The (Single) List Programming Paradigm

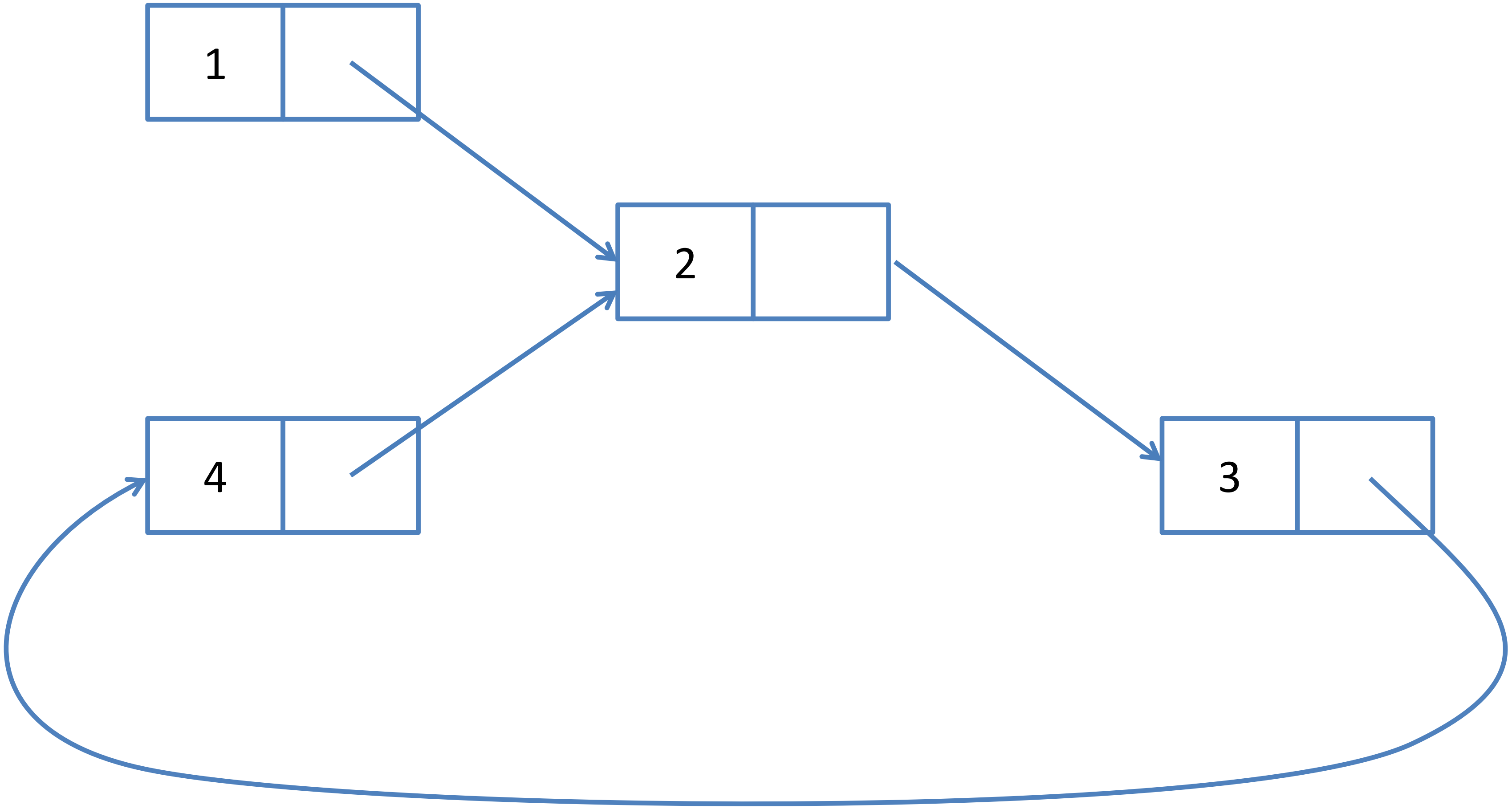
- Recall that a list is either:
  - [ ]   (the empty list)
  - v :: vs   (a value v followed by a *previously constructed list* vs)

- Some examples:

```
let l0 = [];;                  (* length is 0 *)
let l1 = 1::l0;;               (* length is 1 *)
let l2 = 2::l1;;               (* length is 2 *)
let l3 = 3::l2;;               (* length is 3 *)
…
```

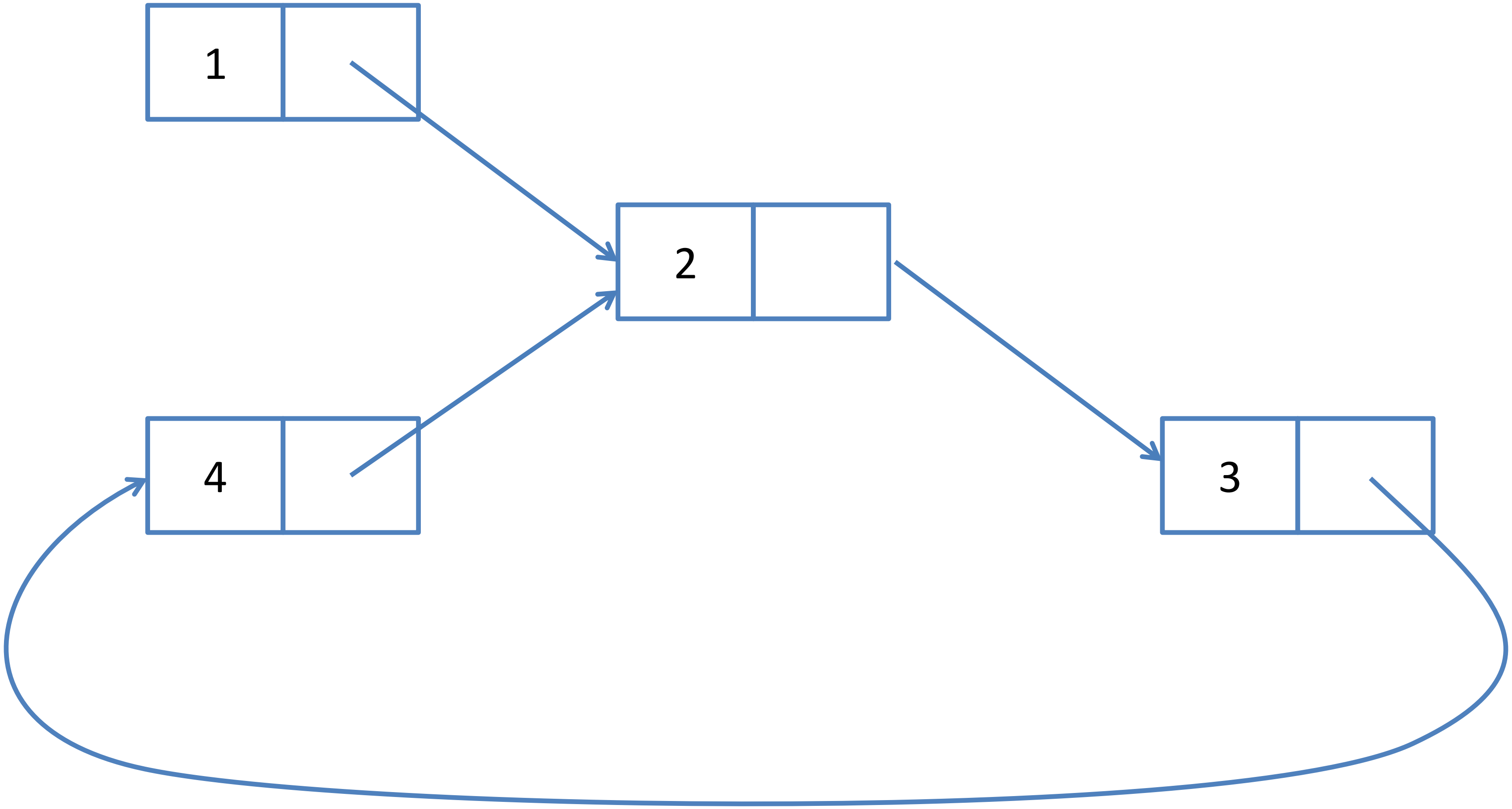
Consider This Picture

- Consider the following picture. How long is the linked structure?
- Can we build a value with type int list to represent it?



Consider This Picture

- How long is it? Infinitely long?
- Can we build a value with type int list to represent it? No!
  - all values with type int list have finite length



The List Type

- Is it a good thing that the type list does not contain any infinitely long lists? Yes!

- A terminating list-processing scheme:

```
let rec f (xs : int list) : int =
  match xs with
    [] -> … do something not recursive …
  | hd::tail -> …  f tail …
```

terminates because f only called recursively on smaller lists

A Loopy Program

```
let rec loop (xs : int list) : int =
  match xs with
     [] -> 0
   | hd::tail -> hd + loop (0::tail)
```

Does this program terminate?

A Loopy Program

```
let rec loop (xs : int list) : int =
  match xs with
    [] -> []
  | hd::tail -> hd + loop (0::tail)
```

Does this program terminate? No! Why not? We call loop recursively on (0::tail). This list is the same size as the original list -- not smaller.

Take-home Message

ML has a *strong type system*

- ML *types say a lot* about the set of values that inhabit them

In this case, the tail of the list is *always* shorter than the whole list

This makes it easy to write functions that terminate; *it would be harder if you had to consider more cases*, such as the case that the tail of a list might loop back on itself. *Moreover OCaml hits you over the head to tell you what the only 2 cases are!*

Note: Just because the list type excludes cyclic structures does not mean that an ML program can't build a cyclic data structure if it wants to. *ML is better than other languages* because it gives you *control* over the values you want to program with, via types!

Rant #2: Imperative lists

- One week from today, ask yourself: Which is easier:
  - Programming with immutable lists in ML?
  - Programming with pointers and mutable ... in C/Java
  - I guarantee you are going ... ML ...
    - there a... ny mor...
    - so many ...

SCORE: OCAML 2, JAVA 0

C: why bother?

Do not believe his lies.

```
let rec xs : int list
= 0::xs
```

let

SCORE: OCAML 1.8, JAVA 0

C: why bother?

Poly-HO!

polymorphic,
higher-order
programming

Some Design & Coding Rules

- Save some software-engineering effort:
  Never write the same code twice.

“Ooh, I get it! I’ll write the code once, copy-paste it somewhere else . . . that way, I didn’t write the same code twice”

  – What’s wrong with that?
    - find and fix a bug in one copy, have to fix in all of them.
    - decide to change the functionality, have to track down all of the places where it gets used.

- Instead, a better practice:
  – factor out the common bits into a reusable procedure.
  – even better: use someone else’s (well-tested, well-documented, and well-maintained) procedure.

Factoring Code in OCaml

Consider these definitions:

```
let rec inc_all (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (hd+1)::(inc_all tl)
```

```
let rec square_all (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (hd*hd)::(square_all tl)
```

Factoring Code in OCaml

Consider these definitions:

```
let rec inc_all (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (hd+1)::(inc_all tl)
```

```
let rec square_all (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (hd*hd)::(square_all tl)
```

The code is almost identical – factor it out!

Factoring Code in OCaml

A *higher-order* function captures the recursion pattern:

```
let rec map (f:int->int) (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)
```

Factoring Code in OCaml

A *higher-order* function captures the recursion pattern:

```
let rec map (f:int->int) (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)
```

Uses of the function:

```
let inc x = x+1
let inc_all xs = map inc xs
```

Factoring Code in OCaml

A *higher-order* function captures the recursion pattern:

```
let rec map (f:int->int) (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)
```

Uses of the function:

```
let inc x = x+1
let inc_all xs = map inc xs

let square y = y*y
let square_all xs = map square xs
```

Writing little functions like inc just so we call map is a pain.

Factoring Code in OCaml

A higher-order function captures the recursion pattern:

```
let rec map (f:int->int) (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl);;
```

Uses of the function:

We can use an *anonymous* function instead.

Originally, Alonzo Church wrote this function using λ instead of **fun**: (λx. x+1) or (λx. x*x)

```
let inc_all xs = map (fun x -> x + 1) xs

let square_all xs = map (fun y -> y * y) xs
```

Another example

```
let rec sum (xs:int list) : int =
  match xs with
  | [] -> 0
  | hd::tl -> hd + (sum tl)


let rec prod (xs:int list) : int =
  match xs with
  | [] -> 1
  | hd::tl -> hd * (prod tl)
```

Goal: Create a function called reduce that when supplied with a few arguments can implement both sum and prod. Define sum2 and prod2 using reduce.

(Try it)

Goal: If you finish early, use map and reduce together to find the sum of the squares of the elements of a list.

(Try it)

Another example

```
let rec sum (xs:int list) : int =
  match xs with
  | [] -> b
  | hd::tl -> hd + (sum tl)


let rec prod (xs:int list) : int =
  match xs with
  | [] -> b
  | hd::tl -> hd * (prod tl)
```

Another example

```
let rec sum (xs:int list) : int =
  match xs with
   | [] -> b
   | hd::tl -> hd OP (RECURSIVE CALL ON tl)


let rec prod (xs:int list) : int =
  match xs with
   | [] -> b
   | hd::tl -> hd OP (RECURSIVE CALL ON tl)
```

Another example

```
let rec sum (xs:int list) : int =
  match xs with
  | [] -> b
  | hd::tl -> f hd (RECURSIVE CALL ON tl)


let rec prod (xs:int list) : int =
  match xs with
  | [] -> b
  | hd::tl -> f hd (RECURSIVE CALL ON tl)
```

A generic reducer

```
let add x y = x + y
let mul x y = x * y

let rec reduce (f:int->int->int) (b:int) (xs:int list) : int =
  match xs with
  | [] -> b
  | hd::tl -> f hd (reduce f b tl)

let sum xs = reduce add 0 xs
let prod xs = reduce mul 1 xs
```

Using Anonymous Functions

```
let rec reduce (f:int->int->int) (b:int) (xs:int list) : int =
  match xs with
  | [] -> b
  | hd::tl -> f hd (reduce f b tl)

let sum xs = reduce (fun x y -> x+y) 0 xs
let prod xs = reduce (fun x y -> x*y) 1 xs
```

Using Anonymous Functions

```
let rec reduce (f:int->int->int) (b:int) (xs:int list) : int =
  match xs with
   | [] -> b
   | hd::tl -> f hd (reduce f b tl)

let sum xs = reduce (fun x y -> x+y) 0 xs
let prod xs = reduce (fun x y -> x*y) 1 xs

let sum_of_squares xs = sum (map (fun x -> x * x) xs)
let pairify xs = map (fun x -> (x,x)) xs
```

Using Anonymous Functions

```
let rec reduce (f:int->int->int) (b:int) (xs:int list) : int =
  match xs with
   | [] -> b
   | hd::tl -> f hd (reduce f b tl)

let sum xs = reduce (+) 0 xs
let prod xs = reduce ( * ) 1 xs

let sum_of_squares xs = sum (map (fun x -> x * x) xs)
let pairify xs = map (fun x -> (x,x)) xs
```

Using Anonymous Functions

```
let rec reduce (f:int->int->int) (b:int) (xs:int list) : int =
  match xs with
   | [] -> b
   | hd::tl -> f hd (reduce f b tl)

let sum xs = reduce (+) 0 xs
let prod xs = reduce (*) 1 xs

let sum_of_squares xs = sum (map (fun x -> x * x) xs)
let pairify xs = map (fun x -> (x,x)) xs
```

wrong

Using Anonymous Functions

```
let rec reduce (f:int->int->int) (b:int) (xs:int list) : int =
  match xs with
  | [] -> b
  | hd::tl -> f hd (reduce f b tl)

let sum xs = reduce (+) 0 xs
let prod xs = reduce (*) 1 xs

let sum_of_squares xs = sum (map (fun x -> x * x) xs)
let pairify xs = map (fun x -> (x,x)) xs
```

wrong -- creates a comment! ug. OCaml -0.1

what does work is: (*)

More on Anonymous Functions

Function declarations:

```
let square x = x*x

let add x y = x+y
```

are *syntactic sugar* for:

```
let square = (fun x -> x*x)

let add = (fun x y -> x+y)
```

In other words, *functions are values* we can bind to a variable, just like 3 or “moo” or true.

Functions are 2nd class no more!

One argument, one result

Simplifying further:

```
let add = (fun x y -> x+y)
```

is shorthand for:

```
let add = (fun x -> (fun y -> x+y))
```

That is, add is a function which:

- when given a value x, *returns a function* (fun y -> x+y) which:
  - when given a value y, returns x+y.

Curried Functions

curry: *verb*

(1) to prepare or flavor with hot-tasting spices

(2) to encode a multi-argument function using nested, higher-order functions.

(1)

(2)

```
fun x -> (fun y -> x+y) (* curried *)
fun x y -> x + y        (* curried *)
fun (x,y) -> x+y        (* uncurried *)
```

Curried Functions

Named after the logician Haskell B. Curry (1950s).

- was trying to find minimal logics that are powerful enough to encode traditional logics.
- much easier to prove something about a logic with 3 connectives than one with 20.
- the ideas translate directly to math (set & category theory) as well as to computer science.
- Actually, Moses Schönfinkel did some of this in 1924
  - thankfully, we don't have to talk about *Schönfinkelled* functions

Curry

Schönfinkel

What's so good about Currying?

In addition to simplifying the language, currying functions so that they only take one argument leads to two major wins:

1. We can *partially apply* a function.
2. We can more easily *compose* functions.

Partial Application

```
let add = (fun x -> (fun y -> x+y))
```

Curried functions allow defs of new, *partially applied* functions:

```
let inc = add 1
```

Equivalent to writing:

```
 let inc = (fun y -> 1+y)
```

which is equivalent to writing:

```
 let inc y = 1+y
```

also:

```
let inc2 = add 2
let inc3 = add 3
```

SIMPLE REASONING ABOUT HIGHER-ORDER FUNCTIONS

Reasoning About Definitions

We can factor this program

```
let square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(square_all tl)
```

into this program:

```
let square_all = map square
```

assuming we already have a definition of map

Reasoning About Definitions

```
let square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(square_all tl)
```

⬇

```
let square_all = map square
```

Goal: Rewrite definitions so my program is simpler, easier to understand, more concise, …

Question: What are the reasoning principles for rewriting programs without breaking them? For reasoning about the behavior of programs? About the equivalence of two programs?

I want some *rules* that never fail.

Simple Equational Reasoning

Rewrite 1 (Function de-sugaring):

```
let f x = body    ==    let f = (fun x -> body)
```

Rewrite 2 (Substitution):

```
(fun x -> ... x ...) arg    ==    ... arg ...
```

if **arg** is a value or, when executed, **will always terminate without effect** and produce a value

roughly: all occurrences of x replaced by arg (though getting this *exactly* right is shockingly difficult)

Rewrite 3 (Eta-expansion):

```
let f = def    ==    let f x = (def) x
```

if f has a function type

chose name x wisely so it does not shadow other names used in def

Using the rules

```
let square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(square_all tl)
```

⬇

```
let square_all = map square
```

Let's use these rules

to prove that these two functions are equivalent

Eliminating the Sugar in Map

```
let rec map f xs =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)
```

Eliminating the Sugar in Map

```
let rec map f xs =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)

let rec map =
  (fun f ->
    (fun xs ->
        match xs with
        | [] -> []
        | hd::tl -> (f hd)::(map f tl)))
```

Consider square_all

```
let rec map =
  (fun f ->
    (fun xs ->
        match xs with
        | [] -> []
        | hd::tl -> (f hd)::(map f tl)))

let square_all =
   map square
```

Substitute map definition into square_all

```
let rec map =
  (fun f ->
    (fun xs ->
        match xs with
        | [] -> []
        | hd::tl -> (f hd)::(map f tl)))

let square_all =
   (fun f ->
       (fun xs ->
           match xs with
           | [] -> []
           | hd::tl -> (f hd)::(map f tl)
       )
   ) square
```

Substitute map definition into square_all

```
let rec map =
  (fun f ->
    (fun xs ->
       match xs with
       | [] -> []
       | hd::tl -> (f hd)::(map f tl)))

let square_all =
   (fun f ->
       (fun xs ->
          match xs with
          | [] -> []
          | hd::tl -> (f hd)::(map f tl)
       )
   ) square
```

Substitute map definition into square_all

```
let rec map =
  (fun f ->
    (fun xs ->
       match xs with
       | [] -> []
       | hd::tl -> (f hd)::(map f tl)))

let square_all =
   (fun f ->
       (fun xs ->
          match xs with
          | [] -> []
          | hd::tl -> (f hd)::(map f tl)
       )
   ) square
```

Substitute Square

```
let rec map =
  (fun f ->
    (fun xs ->
      match xs with
      | [] -> []
      | hd::tl -> (f hd)::(map f tl)))

let square_all =
   (
     (fun xs ->
        match xs with
        | [] -> []
        | hd::tl -> (square hd)::(map square tl)
     )
```

argument square substituted for parameter f

Expanding map square

```
let rec map =
  (fun f ->
    (fun xs ->
       match xs with
       | [] -> []
       | hd::tl -> (f hd)::(map f tl)))

let square_all ys =

      (fun xs ->
         match xs with
         | [] -> []
         | hd::tl -> (square hd)::(map square tl)
      ) ys
```

add argument
via eta-expansion

Expanding map square

```
let rec map =
  (fun f ->
    (fun xs ->
       match xs with
       | [] -> []
       | hd::tl -> (f hd)::(map f tl)))

let square_all ys =



          match ys with
          | [] -> []
          | hd::tl -> (square hd)::(map square tl)
```

substitute again (argument ys for parameter xs)

So Far

```
let rec map f xs =
      match xs with
      | [] -> []
      | hd::tl -> (f hd)::(map f tl)

let square_all xs = map square xs
```

proof by simple rewriting unrolls definition once

```
let square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(map square tl)
```

Next Step

```
let rec map f xs =
      match xs with
      | [] -> []
      | hd::tl -> (f hd)::(map f tl)

let square_all xs = map square xs
```

proof by simple rewriting unrolls definition once

```
let square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(map square tl)
```

proof *by induction* eliminates recursive function map

```
let rec square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(square_all tl)
```

Summary

We saw this:

```
let rec map f xs =
       match xs with
       | [] -> []
       | hd::tl -> (f hd)::(map f tl);;

let square_all = map square
```

Is equivalent to this:

```
let square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(map square tl)
```

Morals of the story:

(1) OCaml's *HOT* (higher-order, typed) functions capture recursion patterns

(2) we can figure out what is going on by *equational reasoning*.

(3) ... but we typically need to do *proofs by induction* to reason about recursive (inductive) functions

POLY-HO!

Here’s an annoying thing

```
let rec map (f:int->int) (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl);;
```

What if I want to increment a list of floats?

Alas, I can’t just call this map. It works on ints!

Here's an annoying thing

```
let rec map (f:int->int) (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl);;
```

What if I want to increment a list of floats?

Alas, I can't just call this map. It works on ints!

```
let rec mapfloat (f:float->float) (xs:float list) :
          float list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(mapfloat f tl);;
```

Turns out

```
let rec map f xs =
  match xs with
   | [] -> []
   | hd::tl -> (f hd)::(map f tl)

let ints = map (fun x -> x + 1) [1; 2; 3; 4]

let floats = map (fun x -> x +. 2.0) [3.1415; 2.718]

let strings = map String.uppercase ["sarah"; "joe"]
```

Type of the undecorated map?

```
let rec map f xs =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)

map : ('a -> 'b) -> 'a list -> 'b list
```

Type of the undecorated map?

```
let rec map f xs =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)

map : ('a -> 'b) -> 'a list -> 'b lis
```

> We often use greek letters like α or β to represent type variables.

Read as:

- for any types 'a and 'b,
- if you give map a function from 'a to 'b,
- it will return a function
  - which when given a list of 'a values
  - returns a list of 'b values.

We can say this explicitly

```
let rec map (f:'a -> 'b) (xs:'a list) : 'b list =
  match xs with
   | [] -> []
   | hd::tl -> (f hd)::(map f tl)

map : ('a -> 'b) -> 'a list -> 'b list
```

The OCaml compiler is smart enough to figure out that this is the *most general* type that you can assign to the code.

(technical term: *principal type*)

We say map is *polymorphic* in the types 'a and 'b – just a fancy way to say map can be used on any types 'a and 'b.

Java generics derived from ML-style polymorphism (but added after the fact and more complicated due to subtyping)

More realistic polymorphic functions

```
let rec merge (lt:'a->'a->bool) (xs:'a list) (ys:'a list) : 'a list =
  match (xs,ys) with
  | ([],_) -> ys
  | (_,[]) -> xs
  | (x::xst, y::yst) ->
      if lt x y then x::(merge lt xst ys)
      else y::(merge lt xs yst)

let rec split (xs:'a list)(ys:'a list)(zs:'a list) : 'a list * 'a list =
  match xs with
  | [] -> (ys, zs)
  | x::rest -> split rest zs (x::ys)

let rec mergesort (lt:'a->'a->bool) (xs:'a list) : 'a list =
  match xs with
  | ([] | _::[]) -> xs
  | _ -> let (first,second) = split xs [] [] in
         merge lt (mergesort lt first) (mergesort lt second)
```

More realistic polymorphic functions

```
mergesort : ('a->'a->bool) -> 'a list -> 'a list

mergesort (<) [3;2;7;1]
  == [1;2;3;7]

mergesort (>) [2; 3; 42]
  == [42 ; 3; 2]

mergesort (fun x y -> String.compare x y < 0) [“Hi”; “Bi”]
  == [“Bi”; “Hi”]

let int_sort = mergesort (<)
let int_sort_down = mergesort (>)
let str_sort = mergesort (fun x y -> String.compare x y < 0)
```

Another Interesting Function

```
let comp f g x = f (g x)

let mystery = comp (add 1) square
```

```
let comp = fun f -> (fun g -> (fun x -> f (g x)))

let mystery = comp (add 1) square
```

```
let mystery =
  (fun f -> (fun g -> (fun x -> f (g x)))) (add 1) square
```

```
let mystery =              fun x -> (add 1) (square x)
```

```
let mystery x = add 1 (square x)
```

Function composition!

```
let comp f g x = f (g x)

let mystery = comp (add 1) square
```

$(f \circ g)(x) = f(g(x))$

mystery = (add 1) $\circ$ square

mystery(x) = (add 1) (square (x))

What is the type of comp?

```
let comp f g x = f (g x)
```

```
let comp (f: 'b->'c) (g: 'a->'b) (x: 'a) : 'c
   = f (g x)
```

```
comp : ('b -> 'c) ->
       ('a -> 'b) ->
       ('a -> 'c)
```

Optimization

What does this program do?

```
map f (map g [x1; x2; …; xn])
```

For each element of the list x1, x2, x3 ... xn, it executes g, creating:

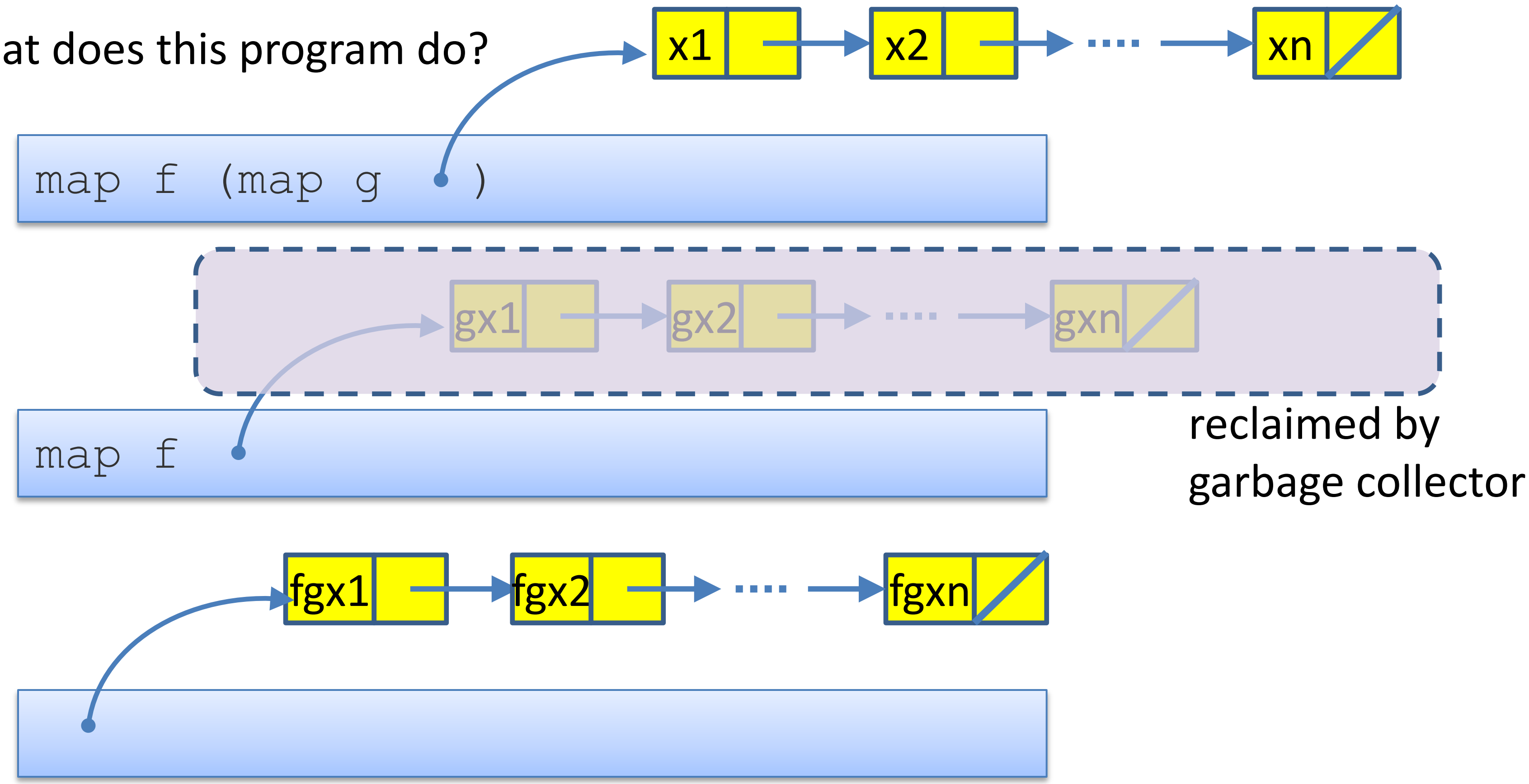
```
map f ([g x1; g x2; …; g xn])
```

Then for each element of the list [g x1, g x2, g x3 ... g xn], it executes f, creating:

```
[f (g x1); f (g x2); …; f (g xn)]
```

Optimization

What does this program do?



Optimization

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For each element of the list x1, x2, x3 ... xn, it executes g, creating:

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map f ([g x1; g x2; …; g xn])
```

Then for each element of the list [g x1, g x2, g x3 ... g xn], it executes f, creating:

```
[f (g x1); f (g x2); …; f (g xn)]
```

Is there a faster way? Yes! (And query optimizers for SQL do it for you.)

```
map (comp f g) [x1; x2; ...; xn]
```

Deforestation

```
map f (map g [x1; x2; …; xn])
```

This kind of optimization has a name:

deforestation

(because it eliminates intermediate lists and, um, trees…)

```
map (comp f g) [x1; x2; ...; xn]
```

How about reduce?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

How about reduce?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What’s the most general ty

Based on the patterns, we know xs must be a ('a list) for some type 'a.

How about reduce?

```
let rec reduce f u (xs: 'a list)  =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

How about reduce?

```
let rec reduce f u (xs: 'a list)  =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What’s the most general type of reduce?

f is called so it must be a function of two arguments.

How about reduce?

```
let rec reduce (f:? -> ? -> ?) u (xs: 'a list)  =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

How about reduce?

```
let rec reduce (f:? -> ? -> ?) u (xs: 'a list)  =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

Furthermore, hd came from xs, so f must take an 'a value as its first argument.

How about reduce?

```
let rec reduce (f:'a -> ? -> ?) u (xs: 'a list)  =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

How about reduce?

```
let rec reduce (f:'a -> ? -> ?) u (xs: 'a list)  =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

The second argument to f must have the same type as the result of reduce. Let's call it 'b.

How about reduce?

```
let rec reduce (f:'a -> 'b -> ?) u (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

The result of f must have the same type as the result of reduce overall: 'b.

How about reduce?

```
let rec reduce (f:'a -> 'b -> 'b) u (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

How about reduce?

```
let rec reduce (f:'a -> 'b -> ?) u (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

If xs is empty, then reduce returns u. So u's type must be 'b.

How about reduce?

```
let rec reduce (f:'a -> 'b -> ?) (u:'b) (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

How about reduce?

```
let rec reduce (f:'a -> 'b -> ?) (u:'b) (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

reduce returns the result of f. So f's result type must be 'b.

How about reduce?

```
let rec reduce (f:'a -> 'b -> 'b) (u:'b) (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What’s the most general type of reduce?

How about reduce?

```
let rec reduce (f:'a -> 'b -> 'b) (u:'b) (xs: 'a list) : 'b =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)
```

What's the most general type of reduce?

```
('a -> 'b -> 'b) -> 'b -> 'a list -> 'b
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery0 = reduce (fun x y -> 1+y) 0
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl);;

let mystery0 = reduce (fun x y -> 1+y) 0;;

let rec mystery0 xs =
  match xs with
  | [] -> 0
  | hd::tl ->
     (fun x y -> 1+y) hd (reduce (fun ...) 0 tl)
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl);;

let mystery0 = reduce (fun x y -> 1+y) 0;;

let rec mystery0 xs =
  match xs with
  | [] -> 0
  | hd::tl ->
     (fun x y -> 1+y) hd (reduce (fun ...) 0 tl)
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl);;

let mystery0 = reduce (fun x y -> 1+y) 0;;

let rec mystery0 xs =
  match xs with
  | [] -> 0
  | hd::tl ->
     (fun y -> 1+y) (reduce (fun ...) 0 tl)
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery0 = reduce (fun x y -> 1+y) 0

let rec mystery0 xs =
  match xs with
  | [] -> 0
  | hd::tl -> 1 + reduce (fun ...) 0 tl
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery0 = reduce (fun x y -> 1+y) 0

let rec mystery0 xs =
  match xs with
  | [] -> 0
  | hd::tl -> 1 + mystery0 tl
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery0 = reduce (fun x y -> 1+y) 0

let rec mystery0 xs =
  match xs with
  | [] -> 0
  | hd::tl -> 1 + mystery0 tl  List Length!
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl);;

let mystery1 = reduce (fun x y -> x::y) []
```

What does this do?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery1 = reduce (fun x y -> x::y) []

let rec mystery1 xs =
  match xs with
  | [] -> []
  | hd::tl -> hd::(mystery1 tl)   Copy!
```

And this one?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery2 g =
   reduce (fun a b -> (g a)::b) []
```

And this one?

```
let rec reduce f u xs =
  match xs with
  | [] -> u
  | hd::tl -> f hd (reduce f u tl)

let mystery2 g =
   reduce (fun a b -> (g a)::b) []

let rec mystery2 g xs =
  match xs with
  | [] -> []
  | hd::tl -> (g hd)::(mystery2 g tl) map!
```

Map and Reduce

```
val map : ('a -> 'b) -> 'a list -> 'b list
```

```
val reduce : ('a -> 'b -> 'b) -> 'b ->  'a list -> 'b
```

We coded map in terms of reduce:

- ie: we showed we can compute map f xs using a call to reduce ? ? ? just by passing the right arguments in place of ? ? ?

Can we code reduce in terms of map?

Map and Reduce

```
val map : ('a -> 'b) -> 'a list -> 'b list
```

```
val reduce : ('a -> 'b -> 'b) -> 'b ->  'a list -> 'b
```

let reduce f u xs =   … map (…) (…) …

(use only: map, f, u, xs; don't use **rec** *)*

reduce (+) 0 [1;2;3] = … map (…) (…) …

Some Other Combinators: List Module

https://caml.inria.fr/pub/docs/manual-ocaml/libref/List.html

```
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a

val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b
```

```
val mapi : (int -> 'a -> 'b) -> 'a list -> 'b list

List.mapi f [a0; ...; an] == [f 0 a0; … ; f n an]
```

```
val map2 : ('a -> 'b -> 'c) -> 'a list -> 'b list -> 'c list

List.map2 f [a0; ...; an] [b0; ...; bn] == [f a0 b0 ; … ; f an bn]
```

```
val iter : ('a -> unit) -> 'a list -> unit

List.iter f [a0; ...; an] == f a0; … ; f an
```

Summary

- Map and reduce are two *higher-order functions* that capture very, very common *recursion patterns*

- Reduce is especially powerful:
  - related to the “visitor pattern” of OO languages like Java.
  - can implement most list-processing functions using it, including things like copy, append, filter, reverse, map, etc.

- We can write clear, terse, reusable code by exploiting:
  - higher-order functions
  - anonymous functions
  - first-class functions
  - polymorphism

Practice Problems

Using map, write a function that takes a list of pairs of integers, and produces a list of the sums of the pairs.

- e.g., list_add [(1,3); (4,2); (3,0)] = [4; 6; 3]
- Write list_add directly using reduce.

Using map, write a function that takes a list of pairs of integers, and produces their quotient if it exists.

- e.g., list_div [(1,3); (4,2); (3,0)] = [Some 0; Some 2; None]
- Write list_div directly using reduce.

Using reduce, write a function that takes a list of optional integers, and filters out all of the None's.

- e.g., filter_none [Some 0; Some 2; None; Some 1] = [0;2;1]
- Why can't we directly use filter? How would you generalize filter so that you can compute filter_none? Alternatively, rig up a solution using filter + map.

Using reduce, write a function to compute the sum of squares of a list of numbers.

- e.g., sum_squares = [3,5,2] = 38