An OCaml definition of OCaml evaluation, or,

Implementing OCaml in OCaml

(Part II)

COS 326
Andrew Appel
Princeton University

slides copyright 2018 David Walker and Andrew Appel
permission granted to reuse these slides for non-commercial educational purposes
Implementing an interpreter:

Components:
- Evaluator for primitive operations
- Substitution
- Recursive evaluation function for expressions
let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e (e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e (x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x)
A MATHEMATICAL DEFINITION* OF OCAML EVALUATION

* it’s a partial definition and this is a big topic; for more, see COS 510
OCaml code can give a language semantics

- **advantage**: it can be executed, so we can try it out
- **advantage**: it is amazingly concise
  - especially compared to what you would have written in Java
- **disadvantage**: it is a little ugly to operate over concrete ML datatypes like “Op_e(e1,Plus,e2)” as opposed to “e1 + e2”

- **big disadvantage**: When you use language X to define the semantics of language Y, you only get a precise definition of Y if you already fully understand the semantics of X. So, when you use OCaml to define the semantics of OCaml, you get a precise definition of OCaml only if you already know the precise definition of OCaml.
PL researchers have developed their own standard notation for writing down how programs execute

- it has a mathematical “feel” that makes PL researchers feel special and gives us goosebumps inside
- it operates over abstract expression syntax like “e1 + e2”
- it is useful to know this notation if you want to read specifications of programming language semantics
  - e.g.: Standard ML (of which OCaml is a descendent) has a formal definition given in this notation (and C, and Java; but not OCaml...)
  - e.g.: most papers in the conference POPL (ACM Principles of Prog. Lang.)
  - Programming languages that have been formally defined this way:
    - Java, Javascript, Rust, C, ML, ...
Our goal is to explain how an expression $e$ evaluates to a value $v$.

In other words, we want to define a mathematical relation between pairs of expressions and values.
We define the “evaluates to” relation using a set of (inductive) rules that allow us to prove that a particular (expression, value) pair is part of the relation.

A rule looks like this:

```
premise 1  premise 2  ...  premise 3

conclusion
```

You read a rule like this:

- “if premise 1 can be proven and premise 2 can be proven and ... and premise n can be proven then conclusion can be proven”

Some rules have no premises

- this means their conclusions are always true
- we call such rules “axioms”
As a rule:

\[
\begin{align*}
\text{e1} \Downarrow v_1 & \quad \text{e2} \Downarrow v_2 & \quad \text{eval\_op}(v_1, \text{op}, v_2) = = v' \\
\text{e1 op e2} \Downarrow v' 
\end{align*}
\]

In English:

“If \text{e1} evaluates to \text{v1} \\
and \text{e2} evaluates to \text{v2} \\
and \text{eval\_op}(\text{v1}, \text{op}, \text{v2}) is equal to \text{v'} \\
then \\
\text{e1 op e2} evaluates to \text{v’}"

In code:

```ml
let rec eval (e:exp) : exp =
    match e with
    | Op_e(e1,op,e2) -> let v1 = eval e1 in
                        let v2 = eval e2 in
                        let v' = eval_op v1 op v2 in
                        v'
```
An example rule

As a rule:

\[
\begin{align*}
    i \in \mathbb{Z} & \quad \Rightarrow \\
    i \downarrow i
\end{align*}
\]

asserts \( i \) is an integer

In English:

“If the expression is an integer value, it evaluates to itself.”

In code:

```ocaml
let rec eval (e:exp) : exp =
    match e with
    | Int_e i -> Int_e i
    ...```

An example rule concerning evaluation

As a rule:

\[
\begin{align*}
e_1 & \Downarrow v_1 & \quad & e_2 [v_1/x] & \Downarrow v_2 \\
\text{let } x = e_1 \text{ in } e_2 & \Downarrow v_2
\end{align*}
\]

In English:

“If \( e_1 \) evaluates to \( v_1 \) (which is a value) and \( e_2 \) with \( v_1 \) substituted for \( x \) evaluates to \( v_2 \) then \( \text{let } x = e_1 \text{ in } e_2 \) evaluates to \( v_2 \).”

In code:

```plaintext
let rec eval (e:exp) : exp =
    match e with
    | Let_e(x,e1,e2) -> let v1 = eval e1 in
                      eval (substitute v1 x e2)
    ...
```
An example rule concerning evaluation

As a rule:

\[ \lambda x.e \Downarrow \lambda x.e \]

typical “lambda” notation for a function with argument x, body e

In English:

“A function value evaluates to itself.”

In code:

```ocaml
let rec eval (e:exp) : exp =
    match e with
    ...
    | Fun_e (x,e) -> Fun_e (x,e)
    ...
```
An example rule concerning evaluation

As a rule:

\[
\begin{align*}
\text{e1} & \Downarrow \lambda x.e \\
\text{e2} & \Downarrow v2 \\
\text{e[v2/x]} & \Downarrow v \\
\text{e1 e2} & \Downarrow v
\end{align*}
\]

In English:

“if e1 evaluates to a function with argument x and body e
and e2 evaluates to a value v2
and e with v2 substituted for x evaluates to v
then e1 applied to e2 evaluates to v”

In code:

```ocaml
let rec eval (e:exp) : exp =
  match e with
  ..
| FunCall_e (e1,e2) ->
    (match eval e1 with
      | Fun_e (x,e) -> eval (substitute (eval e2) x e)
      | ...)
  ...
```
An example rule concerning evaluation

As a rule:
\[
\begin{align*}
  e_1 & \Downarrow \text{rec } f x = e \\
  e_2 & \Downarrow v \\
  e[(\text{rec } f x = e)/f][v/x] & \Downarrow v_2 \\
  e_1 e_2 & \Downarrow v_2
\end{align*}
\]

In English:

“uggh”

In code:

```
let rec eval (e:exp) : exp =
  match e with
  ...
  | (Rec_e (f,x,e)) as f_val ->
    let v = eval e2 in
    eval (substitute f_val f (substitute v x e))
```
Comparison: Code vs. Rules

**Almost** isomorphic:

- one rule per pattern-matching clause
- recursive call to `eval` whenever there is a `⇓` premise in a rule
- what’s the main difference?

**complete eval code:**

```ocaml
let rec eval (e:exp) : exp =
match e with
| Int_e i -> Int_e i
| Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
| Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
| Var_e x -> raise (UnboundVariable x)
| Fun_e (x,e) -> Fun_e (x,e)
| FunCall_e (e1,e2) ->
  (match eval e1
   | Fun_e (x,e) -> eval (Let_e (x,e2,e))
   | _ -> raise TypeError)
| LetRec_e (x,e1,e2) ->
  (Rec_e (f,x,e)) as f_val ->
  let v = eval e2 in
  substitute f_val f (substitute v x e)
```

**complete set of rules:**

<table>
<thead>
<tr>
<th>Rule</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>Int_e i</code> -&gt; <code>Int_e i</code></td>
<td>$i \in \mathbb{Z}$, $i \Downarrow i$</td>
</tr>
<tr>
<td><code>Op_e(e1,op,e2)</code> -&gt; <code>eval_op (eval e1) op (eval e2)</code></td>
<td>$e1 \Downarrow v1$, $e2 \Downarrow v2$, $\text{eval_op} (v1, op, v2) \Downarrow v$</td>
</tr>
<tr>
<td><code>Let_e(x,e1,e2)</code> -&gt; <code>eval (substitute (eval e1) x e2)</code></td>
<td>$e1 \Downarrow v1$, $e2 \Downarrow v2$, $\text{let x = e1 in e2} \Downarrow v2$</td>
</tr>
<tr>
<td><code>Var_e x</code> -&gt; <code>raise (UnboundVariable x)</code></td>
<td>$\lambda x.e \Downarrow \lambda x.e$</td>
</tr>
<tr>
<td><code>Fun_e (x,e)</code> -&gt; <code>Fun_e (x,e)</code></td>
<td>$e1 \Downarrow \lambda x.e$, $e2 \Downarrow v2$, $e[v2/x] \Downarrow v$</td>
</tr>
<tr>
<td><code>FunCall_e (e1,e2)</code> -&gt; <code>eval (Let_e (x,e2,e))</code></td>
<td>$e1 \Downarrow \text{rec f x = e}$, $e2 \Downarrow v2$, $e[e[rec f x = e/f][v2/x]] \Downarrow v3$, $e1 e2 \Downarrow v3$</td>
</tr>
</tbody>
</table>
Comparison: Code vs. Rules

complete eval code:

```ocaml
let rec eval (e:exp) : exp =
  match e with
  | Int_i i -> Int_i i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x)
  | Fun_e (x,e) -> Fun_e (x,e)
  | FunCall_e (e1,e2) ->
    (match eval e1
      | Fun_e (x,e) -> eval (Let_e (x,e2,e))
      | _ -> raise TypeError)
  | LetRec_e (x,e1,e2) ->
    (Rec_e (f,x,e)) as f_val ->
    let v = eval e2 in
    eval (substitute f_val f (substitute v x e))
```

complete set of rules:

\[
\begin{align*}
\text{let } & \ i \epsilon Z \\
\implies & \ i \downarrow \ i \\
\hline
\text{e1} \downarrow v1 & \text{e2} \downarrow v2 & \text{eval_op} (v1, op, v2) = v \\
\hline
\text{e1 op e2} \downarrow v \\
\hline
\text{e1} \downarrow v1 & \text{e2} \downarrow v2 & \text{let } x = e1 \text{ in } e2 \downarrow v2 \\
\hline
\lambda x.e \downarrow \lambda x.e \\
\hline
\text{e1} \downarrow \lambda x.e & \text{e2} \downarrow v2 & \text{e[v2/x]} \downarrow v \\
\hline
\text{e1 e2} \downarrow v \\
\hline
\text{e1} \downarrow \text{rec f x} = e & \text{e2} \downarrow v2 & \text{e[rec f x} = e/f][v2/x] \downarrow v3 \\
\hline
\text{e1 e2} \downarrow v3
\end{align*}
\]

- There’s no formal rule for handling free variables
- No rule for evaluating function calls when a non-function in the caller position
- In general, **no rule when further evaluation is impossible**
  - the rules express the **legal evaluations** and say nothing about what to do in error situations
  - the code handles the error situations by raising exceptions
  - type theorists prove that well-typed programs don’t run into undefined cases
We can reason about OCaml programs using a substitution model.

- integers, bools, strings, chars, and functions are values
- value rule: values evaluate to themselves
- let rule: “let x = e1 in e2”: substitute e1’s value for x into e2
- fun call rule: “(fun x -> e2) e1”: substitute e1’s value for x into e2
- rec call rule: “(rec x = e1) e2”: like fun call rule, but also substitute recursive function for name of function
  - To unwind: substitute (rec x = e1) for x in e1

We can make the evaluation model precise by building an interpreter and using that interpreter as a specification of the language semantics.

We can also specify the evaluation model using a set of inference rules
- more on this in COS 510
Limitations

The substitution model is only a model.

– it does not accurately model all of OCaml’s features
  • I/O, exceptions, mutation, concurrency, ...
  • we can build models of these things, but they aren’t as simple.
  • even substitution is tricky to formalize!
Limitations

The substitution model is only a model.

- it does not accurately model all of OCaml’s features
  - I/O, exceptions, mutation, concurrency, ...
  - we can build models of these things, but they aren’t as simple.
  - even substitution is tricky to formalize!

You can say that again! I got it wrong the first time I tried, in 1932. Fixed the bug by 1934, though.

Alonzo Church, 1903-1995
Princeton Professor, 1929-1967
Limitations

The substitution model is only a model.

- it does not accurately model all of OCaml’s features
  - I/O, exceptions, mutation, concurrency, ...
  - we can build models of these things, but they aren’t as simple.
  - even substitution is tricky to formalize!

It’s useful for reasoning about correctness of algorithms.

- we can use it to formally prove that, for instance:
  - \( \text{map } f \ (\text{map } g \ \text{xs}) = \text{map } (\text{comp } f \ g) \ \text{xs} \)
  - proof: by induction on the length of the list \( \text{xs} \), using the definitions of the substitution model.

- we often model complicated systems (e.g., protocols) using a small functional language and substitution-based evaluation.

It is not useful for reasoning about execution time or space.

- more complex models needed there
Nested Evaluation, aka, “inlining” is a common compiler optimization.

It is also used in theorem provers to reason about equality of expressions.
let g x = 
  let f = fun y -> y + x in 
  let x = 3 in 
  f x
let g x = 
  let f = fun y -> y + x in 
  let x = 3 in 
  f x

g 10
Reasoning about Nested Evaluation

let g x =  
  let f = fun y -> y + x in  
  let x = 3 in  
  f x

let f = fun y -> y + 10 in  
let x = 3 in  
f x

g 10  
-->  
  let f = fun y -> y + 10 in  
  let x = 3 in  
  f x
let g x = 
  let f = fun y -> y + x in 
  let x = 3 in 
  f x

let f = fun y -> y + x in 
let x = 3 in 
((fun y -> y + 10) x)
let g x =
    let f = fun y -> y + x in
    let x = 3 in
    f x

let f = fun y -> y + 10 in
let x = 3 in
(fun y -> y + 10) x

(g 10)
--> let f = fun y -> y + 10 in
    let x = 3 in
    f x
--> let x = 3 in
    (fun y -> y + 10) x
--> (fun y -> y + 10) 3
let g x =
  let f = fun y -> y + x in
  let x = 3 in
  f x

let f = fun y -> y + 10 in
let x = 3 in
f x

let x = 3 in
(fun y -> y + 10) x

(fun y -> y + 10) 3

(3 + 10)

13
let g x = 
let f = fun y -> y + x in 
let x = 3 in 
f x

let g x =
  ( let x = 3 in 
    f x ) [fun y -> y + x / f ]
Reasoning about Nested Evaluation

```
let g x =  
  let f = fun y -> y + x in  
  let x = 3 in  
  f x
```

```
let g x =  
  ( let x = 3 in  
    f x ) [fun y -> y + x / f ]
```

```
let g x =  
  let x = 3 in  
  ((fun y -> y + x) x)
```
Reasoning about Nested Evaluation

```
let g x =
  let f = fun y -> y + x in
  let x = 3 in
  f x
```

Inline

```
let g x =
  ( let x = 3 in
    f x                ) [fun y -> y + x / f ]
```

Substitute

```
let g x =
  let x = 3 in
  ((fun y -> y + x) x)
```

Eval

```
let g x =
  let x = 3 in
  x + x
```
let g x = 
let f = fun y -> y + x in
let x = 3 in
f x

let g x = 
let x = 3 in
x + x
Reasoning about Nested Evaluation

let g x = 
let f = fun y -> y + x in 
let x = 3 in 
f x

let g x = 
let x = 3 in 
x + x

\[ g(10) \rightarrow * \ 13 \]
let g x =
let f = fun y -> y + x in
let x = 3 in
f x

g 10 -->* 13

let g x =
let x = 3 in
x + x

g 10
--> 
let x = 3 in
x + x
Reasoning about Nested Evaluation

```
let g x = 
  let f = fun y -> y + x in 
  let x = 3 in 
  f x
```

```
let g x = 
  let x = 3 in 
  x + x
```

```
g 10 -> 13
```

```
g 10 
  -> 
  let x = 3 in 
  x + x 
  -> 
  3 + 3 
  -> 
  6
```
Reasoning about Nested Evaluation

let g x = 
let f = fun y -> y + x in 
let x = 3 in 
f x

let g x = 
let x = 3 in 
x + x

Our goal in inlining is to make the computation more efficient but to get the same answer!

The transformation is incorrect.
Reasoning about Nested Evaluation

let g x =
let f = fun y -> y + x in
let x = 3 in
f x

let g x =
( let x = 3 in
  f x ) [fun y -> y + x / f ]

let g x =
  let x = 3 in
  ((fun y -> y + x) x)

The x inside the function f was “captured” by the enclosing let. Substitution should be “capture-avoiding”.
let g x =
  let f = fun y -> y + x in
  let x = 3 in
  f x

let g x =
  ( let x = 3 in
    f x ) [fun y -> y + x / f ]

let g x =
  ( let z = 3 in
    f z ) [fun y -> y + x / f ]

let g x =
  let z = 3 in
  (fun y -> y + x) z
Solution: More Generally

\[
(let \ x = e_1 \ in \ e_2) \ [e/y] \ = \ let \ x = e_1' \ in \ e_2'
\]

where

\[
e_1' = e_1 \ [e/y]
\]

\[
e_2' = e_2 \quad \text{if } y = x
\]

\[
e_2' = e_2 \ [e/y] \quad \text{if the free variables of } e \ \text{do not include } x
\]

\[
\quad \text{and if } y \neq x
\]

and otherwise, choose an unused variable \( z \) and

\[
\text{alpha-convert } let \ x = \ldots \ \text{in} \ \ldots \ \text{to} \ let \ z = \ldots \ \text{in} \ \ldots
\]
\[(\text{let } x = e_1 \text{ in } e_2) [e/y] = \text{let } x = e_1' \text{ in } e_2'\]

where
\[e_1' = e_1 [e/y]\]
\[e_2' = e_2 \text{ if } y = x\]
\[e_2' = e_2 [e/y] \text{ if the free variables of } e \text{ do not include } x\]
and if \(y \neq x\)

and otherwise, choose an unused variable \(z\) and
alpha-convert \(\text{let } x = \ldots \text{ in } \ldots\) to \(\text{let } z = \ldots \text{ in } \ldots\)
ASSIGNMENT #4
Two Parts

Part 1: Build your own interpreter

– More features: booleans, pairs, lists, match
– Different model: environment-based vs substitution-based
  • The abstract syntax tree $\text{Fun}_e(\_\_, \_)$ is no longer a value
    – a $\text{Fun}_e$ is not a result of a computation
  • There is one more computation step to do:
    – creation of a closure from a $\text{Fun}_e$ expression

Part 2: Prove facts about programs using equational reasoning

– we have already seen a bit of equational reasoning
  • if $e_1 \rightarrow e_2$ then $e_1 == e_2$
– more in precept and next week
AN ENVIRONMENT MODEL FOR PROGRAM EXECUTION
Consider the following program:

```ocaml
let choose (arg:bool * int * int) : int -> int =
  let (b, x, y) = arg in
  if b then
    (fun n -> n + x)
  else
    (fun n -> n + y)
choose (true, 1, 2)
```
Consider the following program:

```ocaml
let choose (arg : bool * int * int) : int -> int =
  let (b, x, y) = arg in
  if b then
    (fun n -> n + x)
  else
    (fun n -> n + y)

choose (true, 1, 2)
```

Its execution behavior according to the substitution model:
Consider the following program:

```ocaml
let choose (arg:bool * int * int) : int -> int =
  let (b, x, y) = arg in
  if b then
    (fun n -> n + x)
  else
    (fun n -> n + y)

choose (true, 1, 2)
```

Its execution behavior according to the substitution model:

```ocaml
choose (true, 1, 2)
--> 
  let (b, x, y) = (true, 1, 2) in
  if b then (fun n -> n + x)
  else (fun n -> n + y)
```
Consider the following program:

```plaintext
let choose (arg:bool * int * int) : int -> int =
let (b, x, y) = arg in
if b then
  (fun n -> n + x)
else
  (fun n -> n + y)

choose (true, 1, 2)
```

Its execution behavior according to the substitution model:

```plaintext
choose (true, 1, 2)
-->
let (b, x, y) = (true, 1, 2) in
if b then (fun n -> n + x)
else (fun n -> n + y)
-->
if true then (fun n -> n + 1)
else (fun n -> n + 2)
```
Consider the following program:

```ocaml
let choose (arg:bool * int * int) : int -> int =
let (b, x, y) = arg in
if b then
  (fun n -> n + x)
else
  (fun n -> n + y)

choose (true, 1, 2)
```

Its execution behavior according to the substitution model:

```ocaml
choose (true, 1, 2)

--> let (b, x, y) = (true, 1, 2) in
    if b then (fun n -> n + x)
    else (fun n -> n + y)

--> if true then (fun n -> n + 1)
    else (fun n -> n + 2)

--> (fun n -> n + 1)
```
Substitution

How much work does the interpreter have to do?

traverse the entire function body, making a new copy with substituted values

choose (true, 1, 2)

let (b, x, y) = (true, 1, 2) in
if b then (fun n -> n + x)
else (fun n -> n + y)

if true then (fun n -> n + 1)
else (fun n -> n + 2)

(fun n -> n + 1)
Substitution

How much work does the interpreter have to do?

```
choose (true, 1, 2)
```

```
let (b, x, y) = (true, 1, 2) in
if b then (fun n -> n + x)
else (fun n -> n + y)
```

```
if true then (fun n -> n + 1)
else (fun n -> n + 2)
```

```
(fun n -> n + 1)
```

does the interpreter have to do?

traverse the entire function body, making a new copy with substituted values.

traverse the entire function body, making a new copy with substituted values.
Substitution

How much work does the interpreter have to do?

traverse the entire function body, making a new copy with substituted values

traverse the entire function body, making a new copy with substituted values

choose \((true, 1, 2)\)

\[
\text{let } (b, x, y) = (true, 1, 2) \text{ in } \\
\text{if } b \text{ then } (\text{fun } n \rightarrow n + x) \\
\text{else } (\text{fun } n \rightarrow n + y)
\]

\[
\text{if } true \text{ then } (\text{fun } n \rightarrow n + 1) \\
\text{else } (\text{fun } n \rightarrow n + 2)
\]

\[
(\text{fun } n \rightarrow n + 1)
\]
Substitution

How much work does the interpreter have to do?

Every step takes time proportional to the size of the program.

We had to traverse the “else” branch of the if twice, even though we never executed it!
The substitution model of evaluation is *just a model*. It says that we generate new code at each step of a computation. We don’t do that in reality. Too expensive!

The substitution model is good for reasoning about the input-output behavior of a function but doesn’t tell us much about the resources used along the way.

Efficient interpreters use *environments* rather than substitution.

You can think of an environment as *delaying* substitution until it is needed.
An *environment* is a key-value store where the keys are variables and the values are ... programming language values.

**Example:**

\[x -> 1; b -> true; y -> 2]\]

this environment:

- binds 1 to \( x \)
- binds true to \( b \)
- binds 2 to \( y \)
Execution with substitution:

```
let x = 3 in
let b = true in
if b then x else 0
-->
let b = true in
if b then 3 else 0
-->
if true then 3 else 0
-->
3
```

Form of the semantic relation:

```
e1 --> e2
```
Execution with Environment Models

Execution with substitution:

```
let x = 3 in
let b = true in
if b then x else 0
-->
let b = true in
if b then 3 else 0
-->
if true then 3 else 0
-->
3
```

Form of the semantic relation:

```
e1 --> e2
```

Execution with environments:

```
([], let x = 3 in
 let b = true in
 if b then x else 0)
```

Form of the semantic relation:

```
(env1, e1) --> (env2, e2)
```
Execution with Environment Models

Execution with substitution:

\[
\begin{align*}
\text{let } x &= 3 \text{ in} \\
\text{let } b &= \text{true} \text{ in} \\
\text{if } b \text{ then } x \text{ else } 0 \\
\rightarrow \\
\text{let } b &= \text{true} \text{ in} \\
\text{if } b \text{ then } 3 \text{ else } 0 \\
\rightarrow \\
\text{if } \text{true} \text{ then } 3 \text{ else } 0 \\
\rightarrow \\
3
\end{align*}
\]

Execution with environments:

\[
\begin{align*}
([], \text{let } x &= 3 \text{ in} \\
\text{let } b &= \text{true} \text{ in} \\
\text{if } b \text{ then } x \text{ else } 0) \\
\rightarrow \\
([x\rightarrow 3], \text{let } b &= \text{true} \text{ in} \\
\text{if } b \text{ then } x \text{ else } 0)
\end{align*}
\]
Execution with substitution:

```
let x = 3 in
let b = true in
if b then x else 0
-->
let b = true in
if b then 3 else 0
-->
if true then 3 else 0
-->
3
```

Execution with environments:

```
([], let x = 3 in
  let b = true in
  if b then x else 0)
-->
([], let x = 3 in
  let b = true in
  if b then 3 else 0)
-->
([], [x -> 3], let b = true in
  if b then x else 0)
-->
([], [x -> 3; b -> true], if b then x else 0)
```
Execution with substitution:

```
let x = 3 in
let b = true in
if b then x else 0

--> let b = true in
if b then x else 0

--> if true then x else 0

--> 3
```

Execution with environments:

```
([], let x = 3 in
let b = true in
if b then x else 0)

--> ([x->3], let b = true in
if b then x else 0)

--> ([x->3;b->true], if b then x else 0)

--> ([x->3;b->true], if true then x else 0)
```
Execution with substitution:

```plaintext
let x = 3 in
let b = true in
if b then x else 0
-->
let b = true in
if b then 3 else 0
-->
if true then 3 else 0
-->
3
```

Execution with environments:

```plaintext
([], let x = 3 in
 let b = true in
 if b then x else 0)
-->
([x->3], let b = true in
 if b then x else 0)
-->
([x->3;b->true], if b then x else 0)
-->
([x->3;b->true], if true then x else 0)
-->
([x->3;b->true], x)
```
Execution with substitution:

\[
\text{let } x = 3 \text{ in }
\text{let } b = \text{true in }
\text{if } b \text{ then } x \text{ else 0}
\]

\[\rightarrow\]

\[
\text{let } b = \text{true in }
\text{if } b \text{ then } 3 \text{ else 0}
\]

\[\rightarrow\]

\[
\text{if } \text{true then } 3 \text{ else 0}
\]

\[\rightarrow\]

\[
3
\]

Execution with environments:

\[
([], \text{let } x = 3 \text{ in }
\text{let } b = \text{true in }
\text{if } b \text{ then } x \text{ else 0})
\]

\[\rightarrow\]

\[
([x \rightarrow 3], \text{let } b = \text{true in }
\text{if } b \text{ then } x \text{ else 0})
\]

\[\rightarrow\]

\[
([x \rightarrow 3; b \rightarrow \text{true}], \text{if } b \text{ then } x \text{ else 0})
\]

\[\rightarrow\]

\[
([x \rightarrow 3; b \rightarrow \text{true}], \text{if true then } x \text{ else 0})
\]

\[\rightarrow\]

\[
([x \rightarrow 3; b \rightarrow \text{true}], x)
\]

\[\rightarrow\]

\[
([x \rightarrow 3; b \rightarrow \text{true}], 3)
\]
Another Example

([],
  (fun x ->
    let f = fun y -> y + x in
    let x = 3 in
    f x) 10 )
Another Example

([],
 (fun x ->
   let f = fun y -> y + x in
   let x = 3 in
   f x) 10 )

-->  

([x -> 10],
 let f = fun y -> y + x in
 let x = 3 in
 f x )
Another Example

```
([],
  (fun x ->
    let f = fun y -> y + x in
    let x = 3 in
    f x) 10 )

-->

([x -> 10],
 let f = fun y -> y + x in
 let x = 3 in
 f x )

-->

([x -> 10; f -> fun y -> y + x],
 let x = 3 in
 f x )
```
Another Example

```
([],
(fun x ->
    let f = fun y -> y + x in
    let x = 3 in
    f x) 10 )
```

```
([x -> 10],
 let f = fun y -> y + x in
 let x = 3 in
 f x )
```

```
([x -> 10; f -> fun y -> y + x],
 let x = 3 in
 f x )
```

```
([x -> 3; f -> fun y -> y + x],
 f x )
```
Another Example

([],
 (fun x ->
   let f = fun y -> y + x in
   let x = 3 in
   f x) 10 )

--> 

([x -> 10],
 let f = fun y -> y + x in
 let x = 3 in
 f x )

--> 

([x -> 10; f -> fun y -> y + x],
 let x = 3 in
 f x )

--> 

([x -> 3; f -> fun y -> y + x],
 f x )
Another Example

([],
 (fun x ->
   let f = fun y -> y + x in
   let x = 3 in
   f x) 10 )

--> (x -> 10; f -> fun y -> y + x],
 (fun y -> y + x) 10 )

([x -> 10],
 let f = fun y -> y + x in
 let x = 3 in
 f x )

--> ([x -> 10; f -> fun y -> y + x],
 let x = 3 in
 f x )

([x -> 3; f -> fun y -> y + x],
 (fun y -> y + x) x )

--> ([x -> 3; f -> fun y -> y + x],
 (fun y -> y + x) 3 )

([x -> 3; f -> fun y -> y + x],
 f x )
Another Example

([],
  (fun x ->
    let f = fun y -> y + x in
    let x = 3 in
    f x) 10 )

--> 

([x -> 10],
  let f = fun y -> y + x in
  let x = 3 in
  f x )

--> 

([x -> 10; f -> fun y -> y + x],
  let x = 3 in
  f x )

--> 

([x -> 3; f -> fun y -> y + x],
 (fun y -> y + x) x )

--> 

([x -> 3; f -> fun y -> y + x],
 (fun y -> y + x) 3 )

--> 

([x -> 3; f -> fun y -> y + x; y -> 3],
 y + x )
Another Example

\[
([\,], \\
(\text{fun } x \to \\
\text{let } f = \text{fun } y \to y + x \text{ in} \\
\text{let } x = 3 \text{ in} \\
f x) \ 10)
\]

\[
([x \to 10], \\
\text{let } f = \text{fun } y \to y + x \text{ in} \\
\text{let } x = 3 \text{ in} \\
f x)
\]

\[
([x \to 10; f \to \text{fun } y \to y + x], \\
\text{let } x = 3 \text{ in} \\
f x)
\]

\[
([x \to 3; f \to \text{fun } y \to y + x], \\
(f \to \text{fun } y \to y + x) \ x)
\]

\[
([x \to 10], \\
\text{let } f = \text{fun } y \to y + x \text{ in} \\
\text{let } x = 3 \text{ in} \\
f x)
\]

\[
([x \to 3; f \to \text{fun } y \to y + x], \\
(f \to \text{fun } y \to y + x) \ 3)
\]

\[
([x \to 3; f \to \text{fun } y \to y + x], \\
(f \to \text{fun } y \to y + x) \ 3 + 3)
\]

\[
([x \to 3; f \to \text{fun } y \to y + x], \\
\text{let } y = 3 \text{ in} \\
y + x)
\]

\[
([x \to 3; f \to \text{fun } y \to y + x; y \to 3], \\
3 + 3)
\]

\[
([x \to 3; f \to \text{fun } y \to y + x; y \to 3], \\
6)
\]
Recall our Problem with Inlining/Substitution

let g x = let f = fun y -> y + x in let x = 3 in f x

Incorrect Inlining

let g x = let x = 3 in x + x

Incorrect Execution

g 10 -->* 13

(([], (fun x -> let f = fun y -> y + x in let x = 3 in f x) 10 ))

Incorrect Inlining

g 10 -->* 6

(([], ...)) -->* ([...], 6)
Another Example

([],
 (fun x ->
   let f = fun y -> y + x in
   let x = 3 in
   f x) 10 )

--> 

([x -> 10],
 let f = fun y -> y + x in
 let x = 3 in
 f x )

--> 

([x -> 10; f -> fun y -> y + x],
 let x = 3 in
 f x )

--> 

([x -> 3; f -> fun y -> y + x],
 (fun y -> y + x) x )

--> 

([x -> 3; f -> fun y -> y + x],
 (fun y -> y + x) 3 )

--> 

([x -> 3; f -> fun y -> y + x],
 y + x )

--> 

([x -> 3; f -> fun y -> y + x; y -> 3],
 3 + 3 )

--> 

([x -> 3; f -> fun y -> y + x; y -> 3],
 6 )
Functions must carry with them the appropriate environment

A closure is a pair of code and environment

In the environment model, function definitions evaluate to function closures
Another Example

([],
(fun x ->
  let f = fun y -> y + x in
  let x = 3 in
  f x) 10 )
Another Example

```ml
(\[],
 (fun \(x\) ->
   let \(f = \) fun \(y\) -> \(y + x\) in
    let \(x = 3\) in
    \(f\ x\) 10)

-->
(\(x -> 10\],
 let \(f = \) fun \(y\) -> \(y + x\) in
 let \(x = 3\) in
 \(f\ x\))
```
Another Example

```
([],
 (fun x ->
   let f = fun y -> y + x in
   let x = 3 in
   f x) 10 )

-->

([x -> 10],
 let f = fun y -> y + x in
 let x = 3 in
 f x )

-->

([x -> 10; f -> closure [x->10] y = y + x],
 let x = 3 in
 f x )
```
Another Example

```plaintext
([],
 (fun x ->
   let f = fun y -> y + x in
   let x = 3 in
   f x) 10 )

--> 

([x -> 10],
 let f = fun y -> y + x in
 let x = 3 in
 f x )

--> 

([x -> 10; f -> closure [x->10] fun y -> y = y + x],
 let x = 3 in
 f x )

--> 

([x -> 3; f -> closure [x->10] fun y -> y = y + x],]
 f x )
```
Another Example

\[
([], \\
 (\text{fun } x \rightarrow \\
 \quad \text{let } f = \text{fun } y \rightarrow y + x \text{ in} \\
 \quad \text{let } x = 3 \text{ in} \\
 \quad f \ x) \ 10 )
\]

\[
([x \rightarrow 10], \\
 \quad \text{let } f = \text{fun } y \rightarrow y + x \text{ in} \\
 \quad \text{let } x = 3 \text{ in} \\
 \quad f \ x )
\]

\[
([x \rightarrow 10; f \rightarrow \text{closure } [x\rightarrow10] \ y = y + x], \\
 \quad \text{let } x = 3 \text{ in} \\
 \quad f \ x)
\]

\[
([x \rightarrow 3; f \rightarrow \text{closure } [x\rightarrow10] \ y = y + x],[],) \\
 x)
\]
Another Example

```
([],
 (fun x ->
   let f = fun y -> y + x in
   let x = 3 in
   f x) 10 )

-->

([x -> 10],
 let f = fun y -> y + x in
 let x = 3 in
 f x )

-->

([x -> 10; f -> closure [x->10] y = y + x],
 let x = 3 in
 f x )

-->

([x -> 3; f -> closure [x->10] y = y + x],
 (closure [x->10] y = y + x) x )

-->

([x -> 3; f -> closure [x->10] y = y + x],
 (closure [x->10] fun y -> y = y + x) 3 )
```
Another Example

When you call a closure, replace the current environment with the closure’s environment, and bind the parameter to the argument.
Another Example

\[
\begin{align*}
([], & \quad (\text{fun } x \rightarrow \\
& \quad \quad \quad \text{let } f = \text{fun } y \rightarrow y + x \text{ in} \\
& \quad \quad \quad \text{let } x = 3 \text{ in} \\
& \quad \quad \quad f \ x) \ 10) \\
\rightarrow & \quad ([x \rightarrow 10], \\
& \quad \quad \text{let } f = \text{fun } y \rightarrow y + x \text{ in} \\
& \quad \quad \text{let } x = 3 \text{ in} \\
& \quad \quad f \ x) \\
\rightarrow & \quad ([x \rightarrow 10; f \rightarrow \text{closure } [x\rightarrow10] y = y + x], \\
& \quad \quad \text{let } x = 3 \text{ in} \\
& \quad \quad f \ x) \\
\rightarrow & \quad ([x \rightarrow 3; f \rightarrow \text{closure } [x\rightarrow10] y = y + x], \\
& \quad \quad \text{let } x = 3 \text{ in} \\
& \quad \quad 3 + 10) \\
\rightarrow & \quad ([x \rightarrow 3; f \rightarrow \text{closure } [x\rightarrow10] y = y + x],] \\
& \quad \quad f \ x)
\end{align*}
\]
In environment-based interpreter, values are drawn from an environment. This is more efficient than using substitution.

To implement nested, higher-order functions, pair functions with the environment in play when the function is defined.

Pairs of function code & environment are called *closures*.

You have two weeks for assignment #4

- Recommendation: Don't wait until next week to start!