Precept Topics

- Minimum Spanning Trees
- Shortest Paths
- Algorithm Design

Relevant Material

- Book chapters: 4.3 and 4.4

A. RECAP: MSTs and Shortest Paths

Your preceptor will give an overview of the content of this week's lectures.

Note: The focus of this precept is on design questions. *The goal is not to come up with an algorithm from scratch, but to model the problem* in a way our tools can solve (today, MST or shortest-path problems).

Feel free to use this space for notes or as scratch paper.

B. EXERCISE: Dorm Rooms and Routers

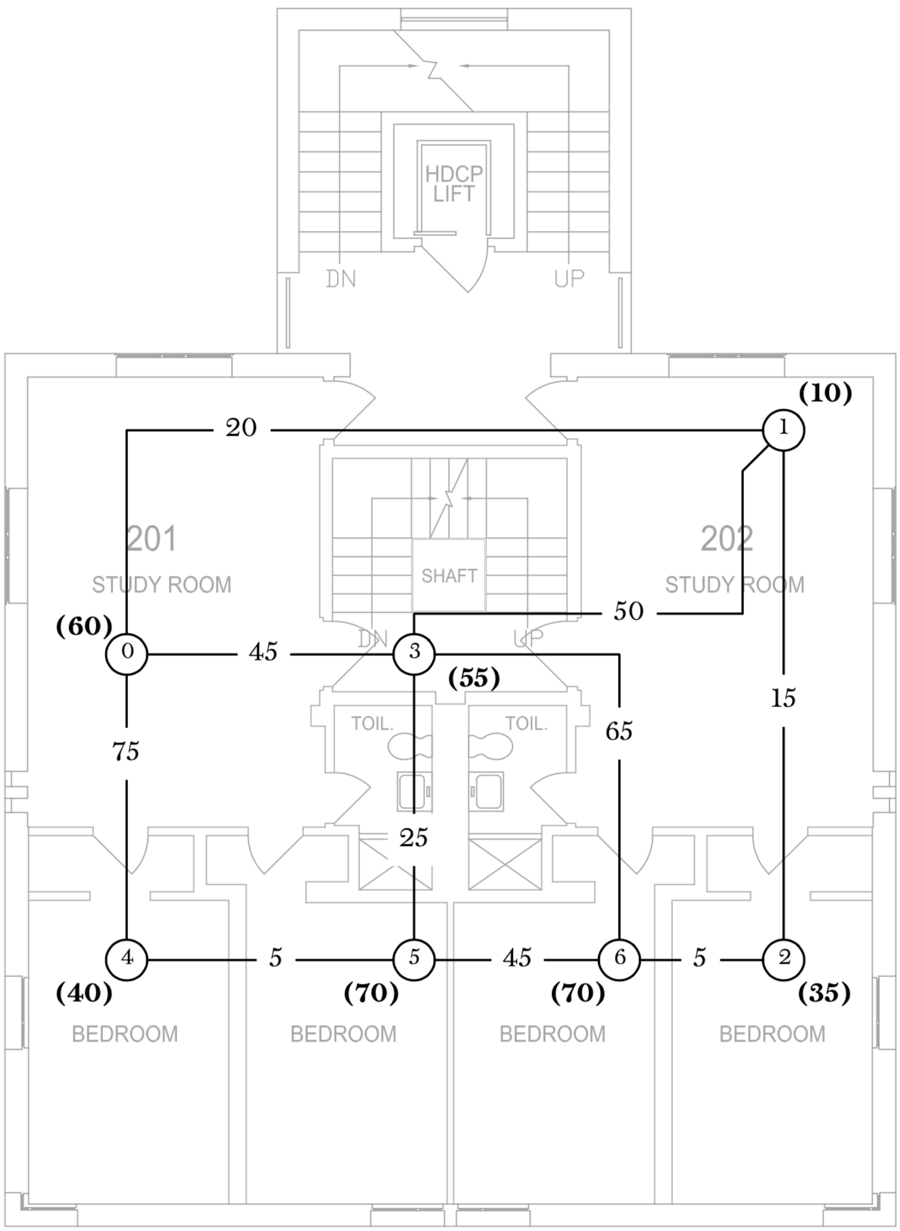
A college has just unveiled a brand-new dorm facility with n rooms. They need to make sure all of them have an internet connection (of course), and are looking for the most cost-effective way to do so. Room number i has internet access if either of the following is true:

- There is a router installed in room i.
- Room i is connected by some fiber path to a room j which has internet access.

Installing a router in room i costs $r_i > 0$, and putting down fiber between rooms i and j costs $f_{ij} > 0$.

The goal of this problem is to determine in which rooms to install a router, and in which pair of rooms to connect together with fiber, so as to minimize the total cost.

Formulate this as a *minimum spanning tree* problem: define a graph $G = (V, E)$ with vertices $V = \{1, 2, \ldots, n\}$ and edges/edge weights that depend on r_i and f_{ij}. You may use the example below to test your formulation.

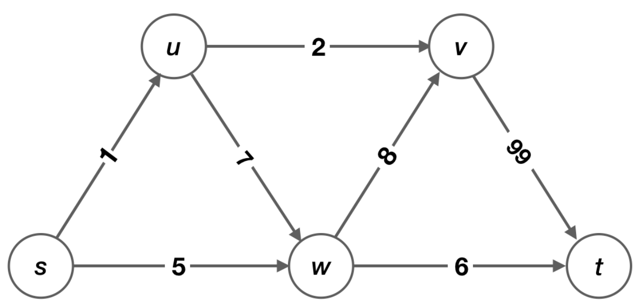


This instance contains 7 dorm rooms and 10 possible connections. The router installation costs are indicated in bold and parentheses; the fiber costs are given on the edges.

C. EXERCISE: Shortest Teleport Path

Given an edge-weighted digraph G with non-negative edge weights, a source vertex s and a destination vertex t, find a shortest path from s to t where you are permitted to teleport across one edge for free. That is, the weight of a path is the sum of the weights of all but the largest edge weights in the path.

For example, in the edge-weighted digraph below, the shortest path from s to t is $s \to w \to t$ (with weight 11) but the shortest teleport path is $s \to u \to v \to t$ (with weight $102 - 99 = 3$).



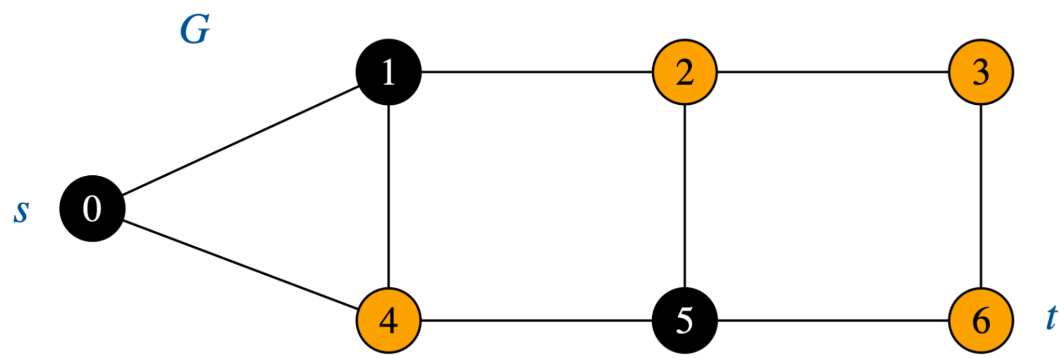
A full solution should run in $O(E \log V)$ time and $O(V)$ extra space.

D. EXERCISE: Shortest Tiger Path (Spring ‘23 Final)

Consider a graph G in which each vertex is colored black or orange. A *tiger path* is a path that contains exactly one edge whose endpoints have opposite colors.

Our goal is to solve the *shortest tiger path problem*: given an undirected graph G and two vertices s and t, find a tiger path between s and t that uses the fewest edges (or report that no such path exists).

For example, the shortest path between $s = 0$ and $t = 6$ in the graph below is $0 \to 4 \to 5 \to 6$, but it is not a tiger path; the shortest tiger path is $0 \to 1 \to 2 \to 3 \to 6$.



Formulate the shortest tiger path problem as a traditional (unweighted) shortest path problem in a directed graph. Specifically, define a digraph G , source s , and destination t such that the length of the shortest path from s to t in G is always equal to the length of the shortest tiger path between s and t in G. For simplicity, you may assume that s is black and t is orange.

For full credit, the number of vertices in G must be $\Theta(V)$ and the number of edges must be $\Theta(E)$, where V and E are the number of vertices and edges in G, respectively.

Challenge Problem (<u>optional</u>): Prove *formally* that the following conditions on an n-vertex undirected graph T are equivalent:

1. T is acyclic and connected;
2. T is maximal among all n-vertex acyclic graphs (i.e., adding an edge creates a cycle);
3. T is minimal among all n-vertex connected graphs (i.e., removing any edge disconnects it).

(Therefore, the mathematical definition of a *tree* is any – equivalently, all – of the above.)