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2.1 Elementary Sorts

- *rules of the game*
- *selection sort*
- *insertion sort*
- *binary search*

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- *rules of the game*
- *selection sort*
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- *binary search*

Sorting problem

Problem. Given an array of n elements, rearrange in ascending order by key.

element → (points to the row Potter, Harry, Gryffindor, 1998)

key → (points to the Last cell Malfoy)

Last ▾	First	House	Year
Longbottom	Neville	Gryffindor	1998
Weasley	Ron	Gryffindor	1998
Abbott	Hannah	Hufflepuff	1998
Potter	Harry	Gryffindor	1998
Chang	Cho	Ravenclaw	1997
Granger	Hermione	Gryffindor	1998
Malfoy	Draco	Slytherin	1998
Diggory	Cedric	Hufflepuff	1996
Weasley	Ginny	Gryffindor	1999
Parkinson	Pansy	Slytherin	1998

sorting hat

Sorting problem

Problem. Given an array of n elements, rearrange in ascending order by key.

Last ▾	First	House	Year
Abbott	Hannah	Hufflepuff	1998
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Longbottom	Neville	Gryffindor	1998
Malfoy	Draco	Slytherin	1998
Parkinson	Pansy	Slytherin	1998
Potter	Harry	Gryffindor	1998
Weasley	Ron	Gryffindor	1998
Weasley	Ginny	Gryffindor	1999

key → (Malfoy)

element → (Potter row)

sorted by key ↑

sorting hat

Sorting problem

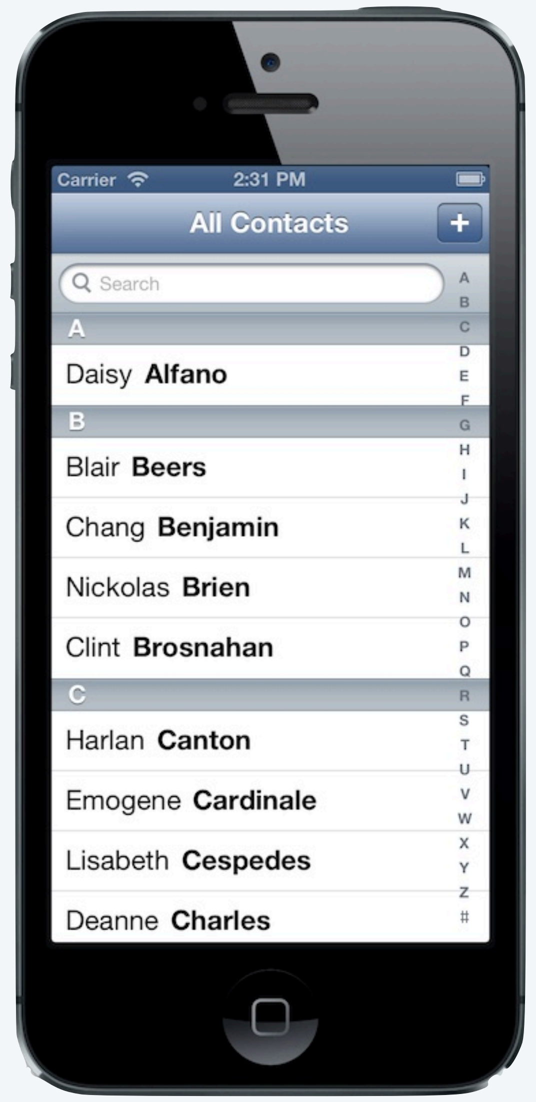
Sorting is a well-defined problem if there is a binary relation ≤ that satisfies:

- Totality: either $v \leq w$ or $w \leq v$ or both.
- Transitivity: if both $v \leq w$ and $w \leq x$, then $v \leq x$.

← *mathematically, a "weak order"*

Examples.

chronological order



alphabetical order

No.	Video name	Views (billions)
1.	"Baby Shark Dance"[3]	10.15
2.	"Despacito"[6]	7.73
3.	"Johny Johny Yes Papa"[12]	6.15
4.	"Shape of You"[13]	5.61
5.	"See You Again"[15]	5.41

numerical order (descending)

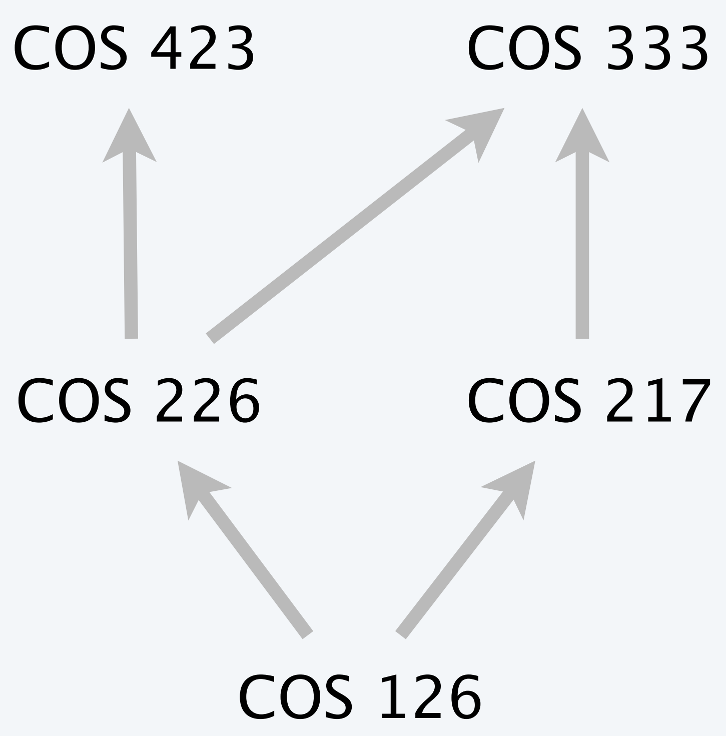
Sorting problem

Sorting is a well-defined problem if there is a binary relation ≤ that satisfies:

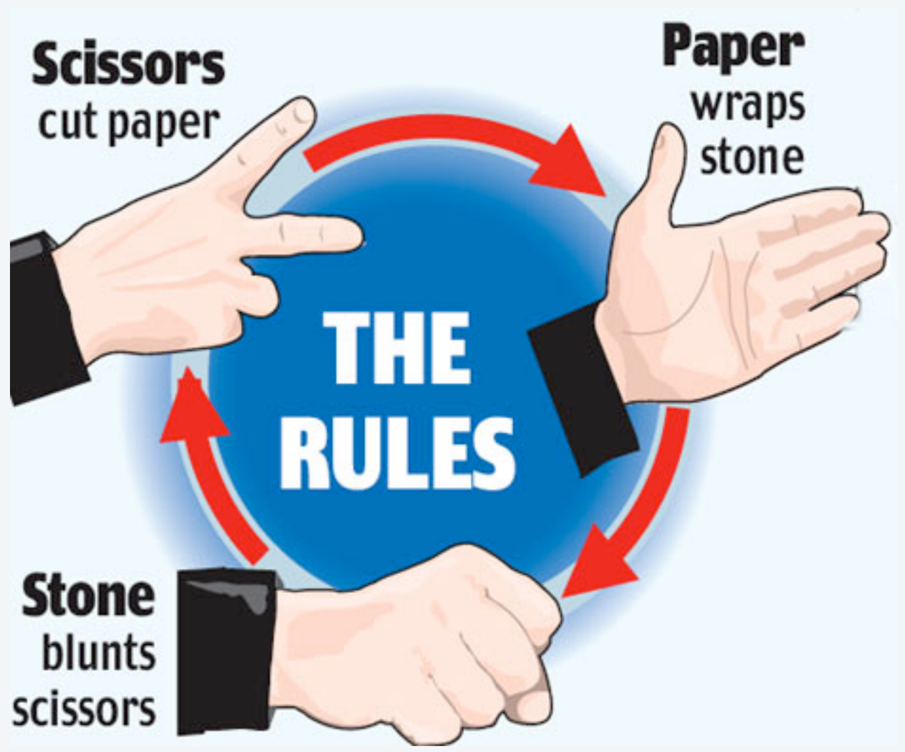
- Totality: either $v \le w$ or $w \le v$ or both.
- Transitivity: if both $v \le w$ and $w \le x$, then $v \le x$.

← *mathematically, a "weak order"*

Non-examples.



course prerequisites
(violates totality)



Ro-sham-bo order
(violates transitivity)

```
~/cos226/sort> jshell
Math.sqrt(-1.0) <= Math.sqrt(-1.0);
false
```

the <= operator for double
(irreflexive, which violates totality)

Sample sort clients

Goal. General-purpose sorting function.

Ex 1. Sort strings in lexicographic order. ← *alphabetical order, using Unicode character ordering*

```
public class StringSorter {
   public static void main(String[] args) {
      String[] a = StdIn.readAllStrings();
      Insertion.sort(a);
      for (int i = 0; i < a.length; i++)
         StdOut.println(a[i]);
   }
}
```

```
~/cos226/sort> more words3.txt
bed bug dad yet zoo ... all bad yes

~/cos226/sort> java StringSorter < words3.txt
all bad bed bug dad ... yes yet zoo
[suppressing newlines]
```

Sample sort clients

Goal. General-purpose sorting function.

Ex 2. Sort real numbers in numerical order (ascending).

```
public class Experiment {
   public static void main(String[] args) {
      int n = Integer.parseInt(args[0]);
      Double[] a = new Double[n];
      for (int i = 0; i < n; i++)
         a[i] = StdRandom.uniformDouble();
      Insertion.sort(a);
      for (int i = 0; i < n; i++)
         StdOut.println(a[i]);
   }
}
```

```
~/cos226/sort> java Experiment 10
0.08614716385210452
0.09054270895414829
0.10708746304898642
0.21166190071646818
0.363292849257276
0.460954145685913
0.5340026311350087
0.7216129793703496
0.9003500354411443
0.9293994908845686
```

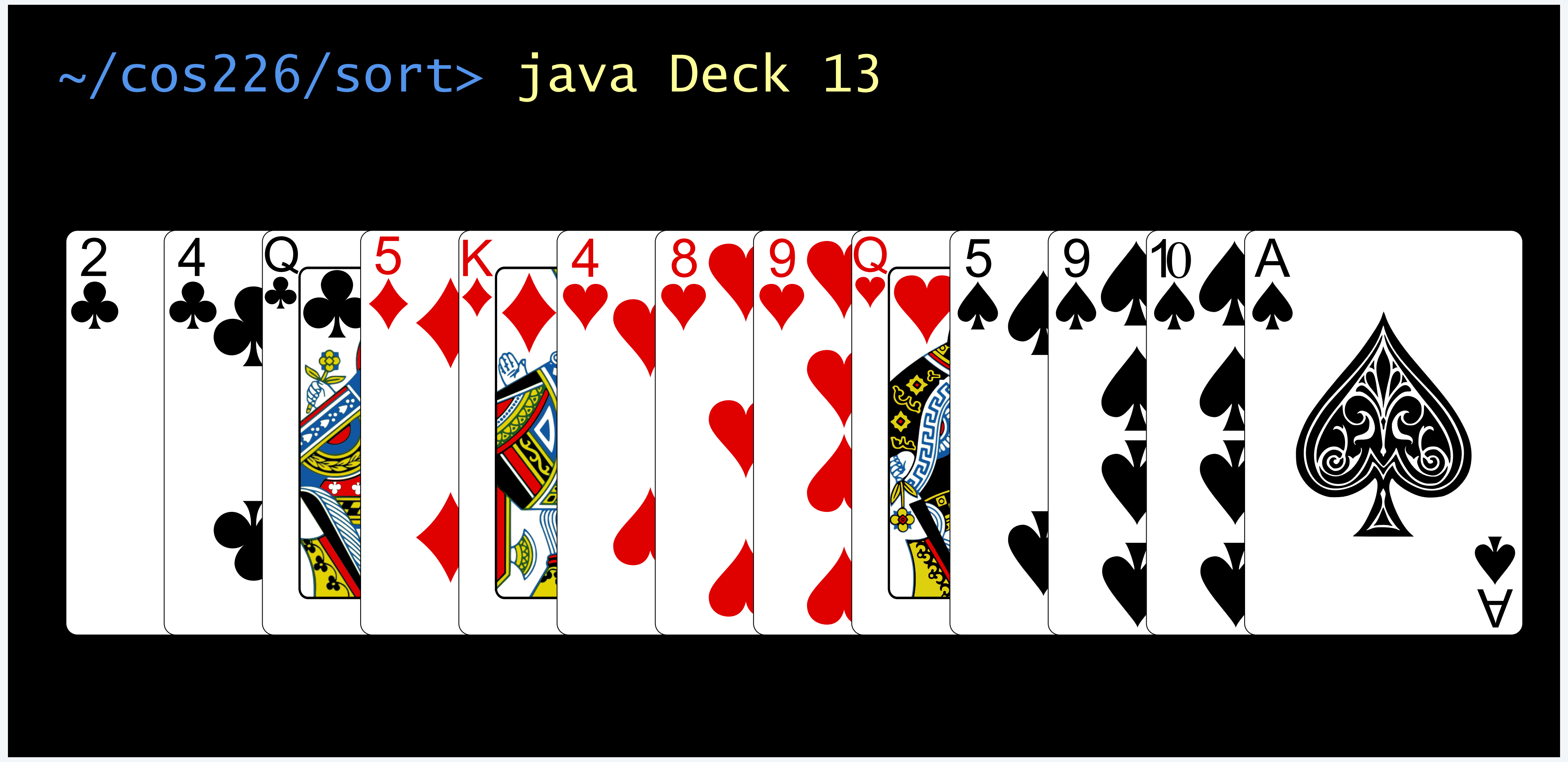
Sample sort clients

Goal. General-purpose sorting function.

Ex 3. Sort playing cards by suit and rank.

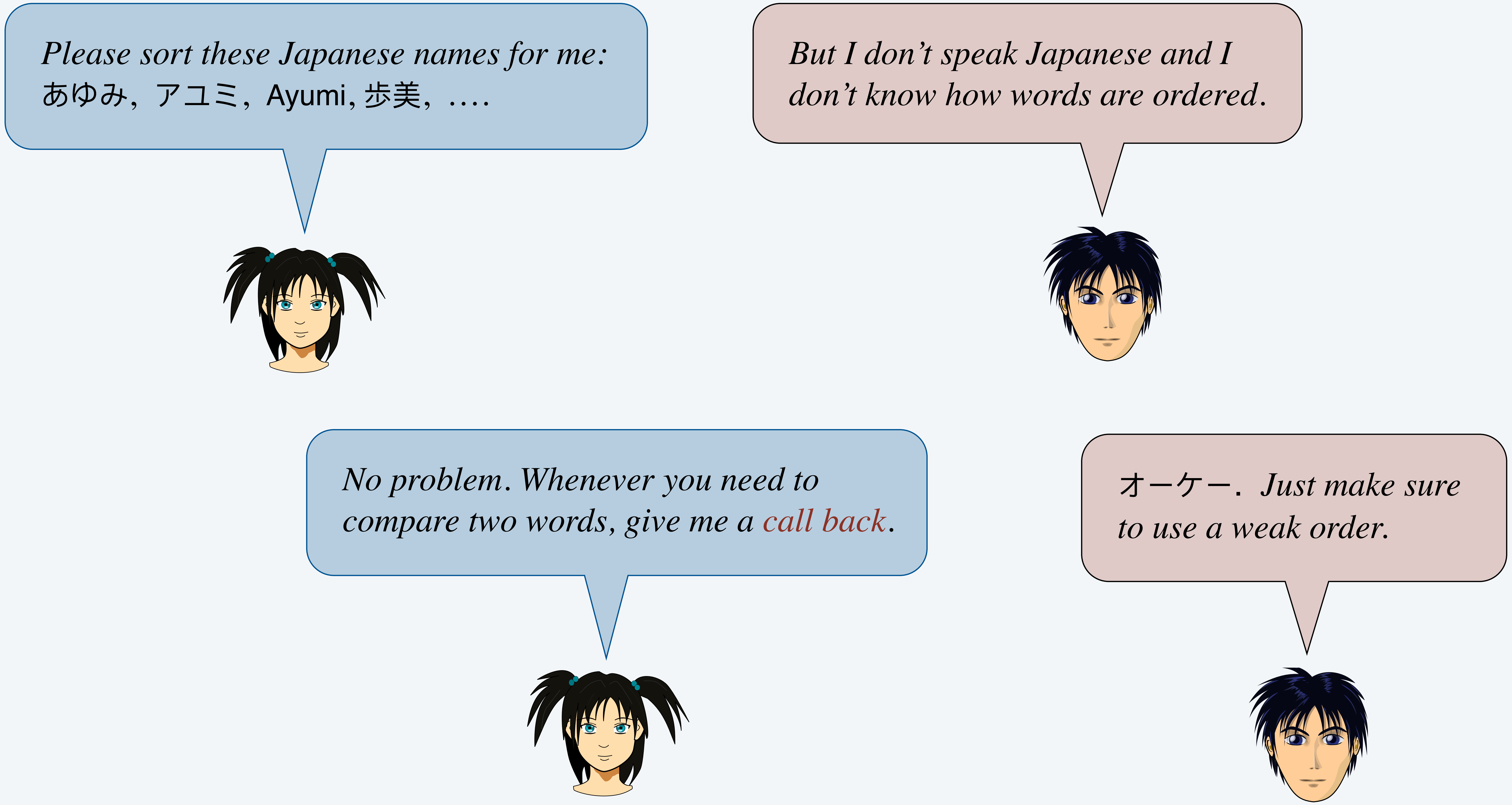
```
public class Deck {
   ...

   public static void main(String[] args) {
      int n = Integer.parseInt(args[0]);
      PlayingCard[] cards = deal(n);
      Insertion.sort(cards);
      draw(cards);
   }
}
```



How can a single function sort any type of data?

Goal. General-purpose sorting function.



Callbacks

Goal. General-purpose sorting function.

Solution. Callback = reference to executable code passed to other code and later executed.

- Client passes array of objects to `sort()` function.
- The `sort()` function calls object's `compareTo()` method as needed.

in effect, client passes `compareTo()` *method to* `sort()` *function; the callback occurs when* `sort()` *invokes* `compareTo()`

Implementing callbacks.

- Java: interfaces.
- Python, ML, Javascript: first-class functions.
- C#: delegates.
- C: function pointers.
- C++: class-type functors.

Review: Java interfaces

Interface. A set of related methods that define some behavior (partial API) for a class.

interface (java.lang.Comparable)

```
public interface Comparable<Item> {
   public int compareTo(Item that);
}
```

contract: method with this signature (and prescribed behavior)

Class that implements interface. Must implement all interface methods.

```
public class String implements Comparable<String> {
    ...

    public int compareTo(String that) {
        ...
    }

}
```

class promises to honor the contract

class abides by the contract

Callbacks in Java: roadmap

client (StringSorter.java)

```
public class StringSorter {
   public static void main(String[] args) {
      String[] a = StdIn.readAllStrings();
      Insertion.sort(a);
      ...
   }
}
```

interface (Comparable.java)

```
public interface Comparable<Item> {
   int compareTo(Item that);
}
```

String[] is a subtype of Comparable[]

sort implementation (Insertion.java)

```
public class Insertion {
   public static void sort(Comparable[] a) {
      ...
      if (a[i].compareTo(a[j]) < 0)
      ...
   }
}
```

data type implementation (String.java)

```
public class String implements Comparable<String> {
   ...
   public int compareTo(String that) {
      ...
   }

}
```

callback

key point: sorting code does not depend upon type of data to be sorted

Suppose that the Java architects left out `implements Comparable<String>` in the class declaration for `String`. What would be the effect?

A. Compile-time error in `String.java`.

B. Compile-time error in `StringSorter.java`.

C. Compile-time error in `Insertion.java`.

D. Run-time exception in `Insertion.java`.

Comparable API

Implement `compareTo()` so that `v.compareTo(w)`

- Returns a negative integer if `v` is less than `w`.
- Returns a positive integer if `v` is greater than `w`.
- Returns zero if `v` is equal to `w`.
- Throws an exception if incompatible types (or either is `null`).

API requirement: the binary relation `v.compareTo(w) <= 0` *is a weak order*

v is less than w (return negative integer)

v is equal to w (return 0)

v is greater than w (return positive integer)

Built-in comparable types. `Integer`, `Double`, `String`, `java.util.Date`, …

User-defined comparable types. Implement the `Comparable` interface.

Implementing the Comparable interface

Date data type. Simplified version of `java.util.Date`.

```
public class Date implements Comparable<Date> {
   private final int month, day, year;

   public Date(int m, int d, int y) {
      month = m;
      day   = d;
      year  = y;
   }

   public int compareTo(Date that) {
      if (this.year  < that.year ) return -1;
      if (this.year  > that.year ) return +1;
      if (this.month < that.month) return -1;
      if (this.month > that.month) return +1;
      if (this.day   < that.day  ) return -1;
      if (this.day   > that.day  ) return +1;
      return 0;
   }

}
```

can compare `Date` *objects only to other* `Date` *objects*

https://algs4.cs.princeton.edu/12oop/Date.java.html

https://algs4.cs.princeton.edu

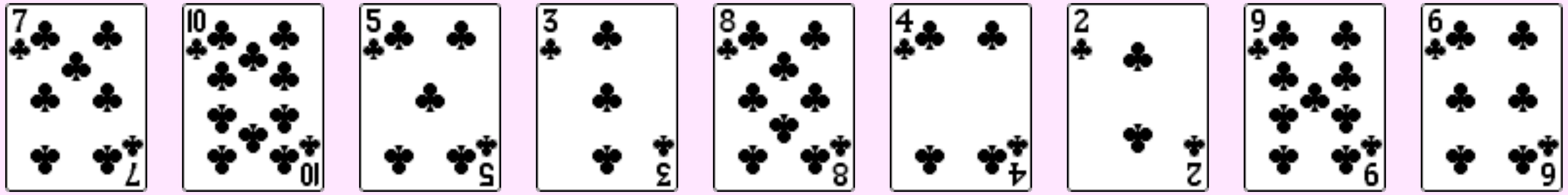
2.1 Elementary Sorts

- rules of the game
- *selection sort*
- insertion sort
- binary search

Selection sort demo

Algorithm. For each index i from 0 to $n - 1$:

- Find index *min* of smallest remaining element.
- Swap elements at indices i and *min*.



initial array

Selection sort: visualization

Visualization. Sort vertical bars by length.

https://www.toptal.com/developers/sorting-algorithms/selection-sort

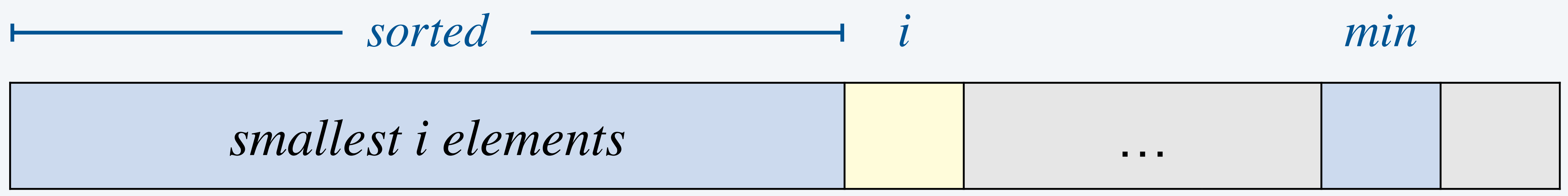
Selection sort invariants

Algorithm. For each index i from 0 to $n-1$:

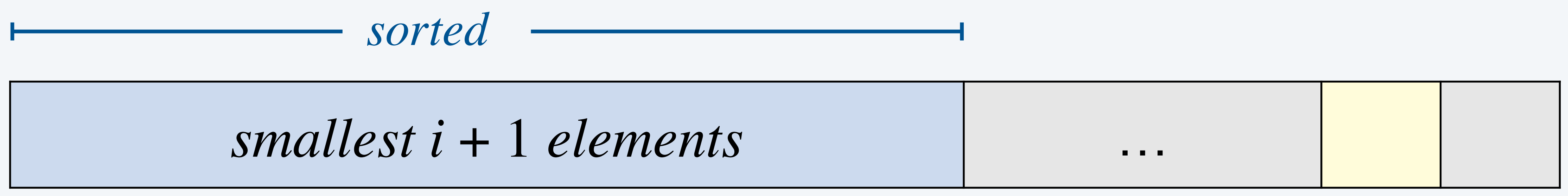
- Find index min of smallest remaining element.
- Swap elements at indices i and min.

Invariants.

before iteration i



after iteration i



Two useful sorting primitives (and a cost model)

Helper functions. Refer to data only through compares and exchanges. ← *e.g., no calls to* `equals()`

use as our cost model for sorting

Compare. Is item `v` less than item `w` ?

```
private static boolean less(Comparable v, Comparable w) {
   return v.compareTo(w) < 0;
}
```

`less("aardvark", "zebra")` *returns* `true`

polymorphic method call

use interface type as argument
⇒ method works for all subtypes

Exchange. Swap array entries `a[i]` and `a[j]`.

```
private static void exch(Object[] a, int i, int j) {
   Object swap = a[i];
   a[i] = a[j];
   a[j] = swap;
}
```

Java arrays are "covariant"
(e.g., `String[]` *is a subtype of* `Object[]`*)*

Selection sort: Java implementation

```java
public class Selection {

   public static void sort(Comparable[] a) {
      int n = a.length;
      for (int i = 0; i < n; i++)
         int min = i;
         for (int j = i+1; j < n; j++)
            if (less(a[j], a[min]))
               min = j;
         exch(a, i, min);
   }

   private static boolean less(Comparable v, Comparable w) {
      /* see previous slide */
   }

   private static void exch(Object[] a, int i, int j) {
      /* see previous slide */
   }

}
```

https://algs4.cs.princeton.edu/21elementary/Selection.java.html

Elementary sorts: quiz 2

How many compares to selection sort an array of n distinct items in reverse order?

A. $\sim n$

B. $\sim 1/4\, n^2$

C. $\sim 1/2\, n^2$

D. $\sim n^2$

Selection sort: mathematical analysis

Proposition. Selection sort makes $(n-1) + (n-2) + \ldots + 1 + 0 \sim \frac{1}{2} n^2$ compares and n exchanges to sort any array of n items.

i	min	0	1	2	3	4	5	6	7	8	9	10
		S	O	R	T	E	X	A	M	P	L	E
0	6	S	O	R	T	E	X	A	M	P	L	E
1	4	A	O	R	T	E	X	S	M	P	L	E
2	10	A	E	R	T	O	X	S	M	P	L	E
3	9	A	E	E	T	O	X	S	M	P	L	R
4	7	A	E	E	L	O	X	S	M	P	T	R
5	7	A	E	E	L	M	X	S	O	P	T	R
6	8	A	E	E	L	M	O	S	X	P	T	R
7	10	A	E	E	L	M	O	P	X	S	T	R
8	8	A	E	E	L	M	O	P	R	S	T	X
9	9	A	E	E	L	M	O	P	R	S	T	X
10	10	A	E	E	L	M	O	P	R	S	T	X
		A	E	E	L	M	O	P	R	S	T	X

entries in black are examined to find the minimum

entries in red are a[min]

entries in gray are in final position

Running time insensitive to input. $\Theta(n^2)$ compares. ← *even if input array is sorted*

Data movement is minimal. $\Theta(n)$ exchanges.

In place. $\Theta(1)$ extra space.

https://algs4.cs.princeton.edu

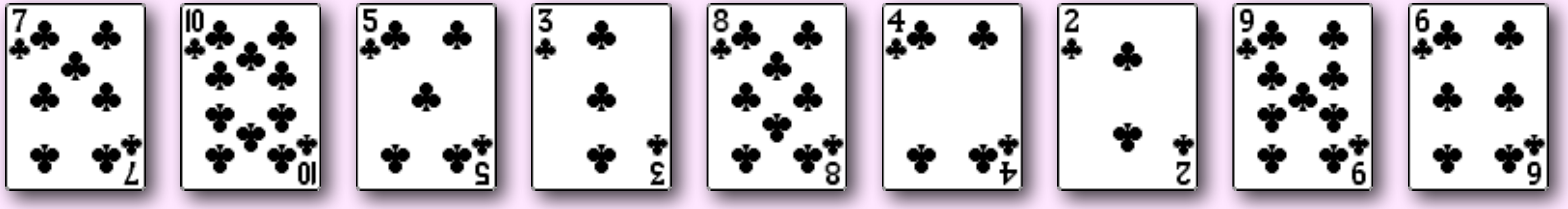
2.1 Elementary Sorts

- rules of the game
- selection sort
- **insertion sort**
- binary search

Insertion sort demo

Algorithm. For each index $i = 0$ to $n - 1$:

- Let x be the element at index i.
- Repeatedly exchange x with each larger element to its immediate left.



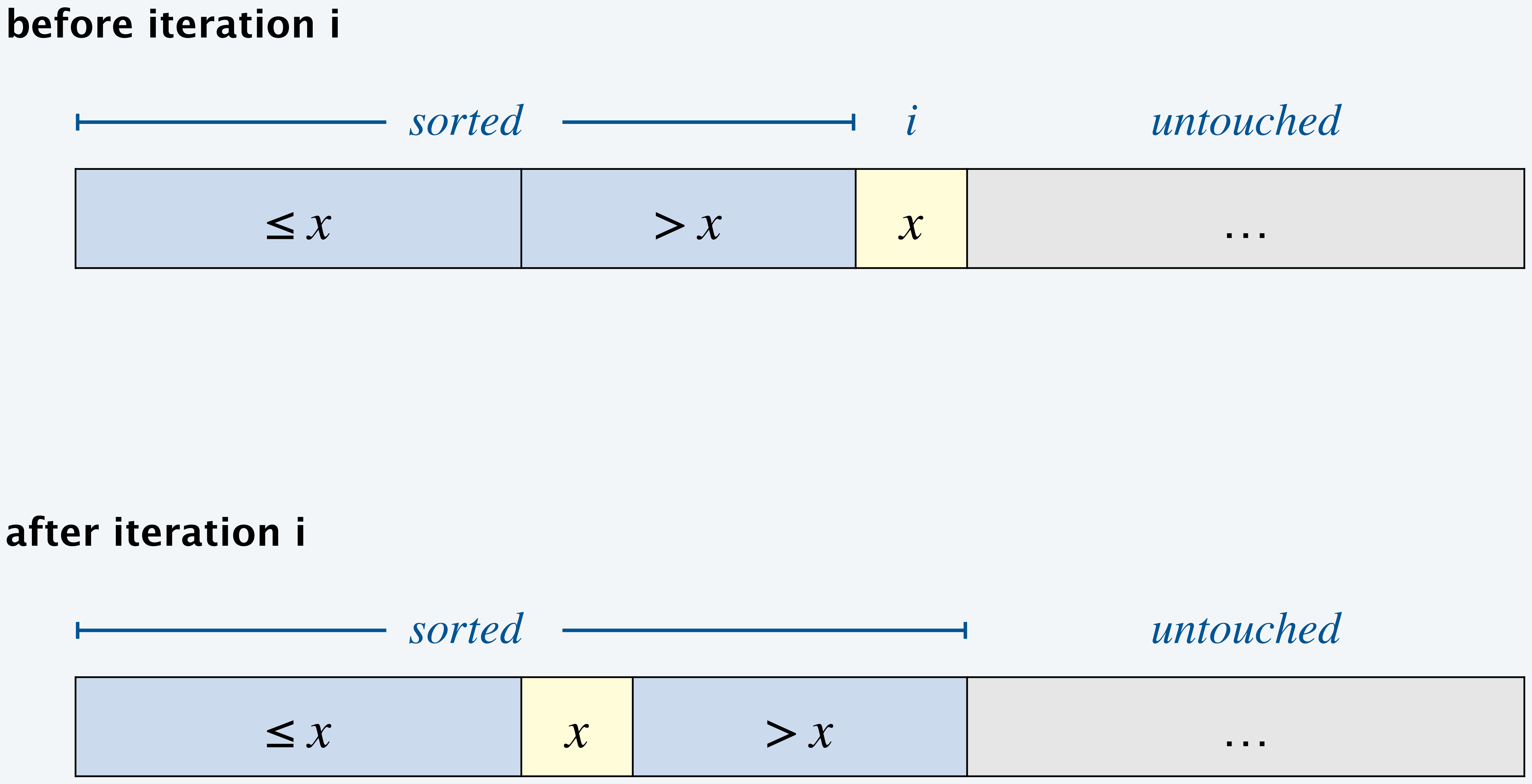
initial array

Insertion sort invariants

Algorithm. For each index $i = 0$ to $n - 1$:

- Let x be the element at index i.
- Repeatedly exchange x with each larger element to its immediate left.

Invariants.



Insertion sort: Java implementation

```
public class Insertion {

   public static void sort(Comparable[] a) {
      int n = a.length;
      for (int i = 0; i < n; i++)
         for (int j = i; j > 0; j--)
            if (less(a[j], a[j-1]))
               exch(a, j, j-1);
            else break;
   }

   private static boolean less(Comparable v, Comparable w) {
      /* as before */
   }

   private static void exch(Object[] a, int i, int j) {
      /* as before */
   }

}
```

https://algs4.cs.princeton.edu/21elementary/Insertion.java.html

Elementary sorts: quiz 4

How many compares to insertion sort an array of n distinct keys in reverse order?

A. $\sim n$

B. $\sim 1/4 \; n^2$

C. $\sim 1/2 \; n^2$

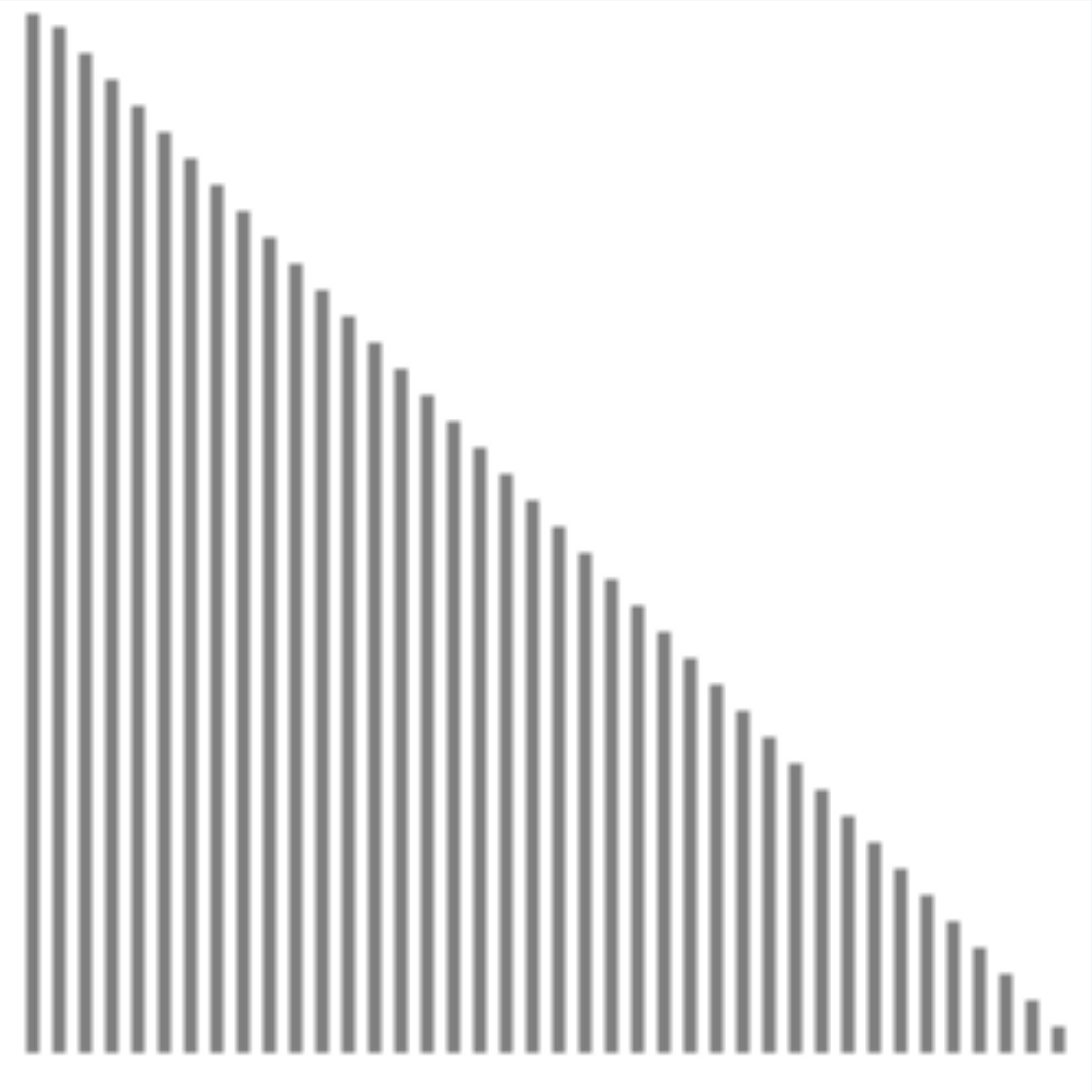
D. $\sim n^2$

Insertion sort: running time analysis

Worst case. Insertion sort makes $\sim \frac{1}{2} n^2$ compares and $\sim \frac{1}{2} n^2$ exchanges to sort an array of n distinct keys in reverse order.

Pf. Exactly i compares and exchanges in iteration i.

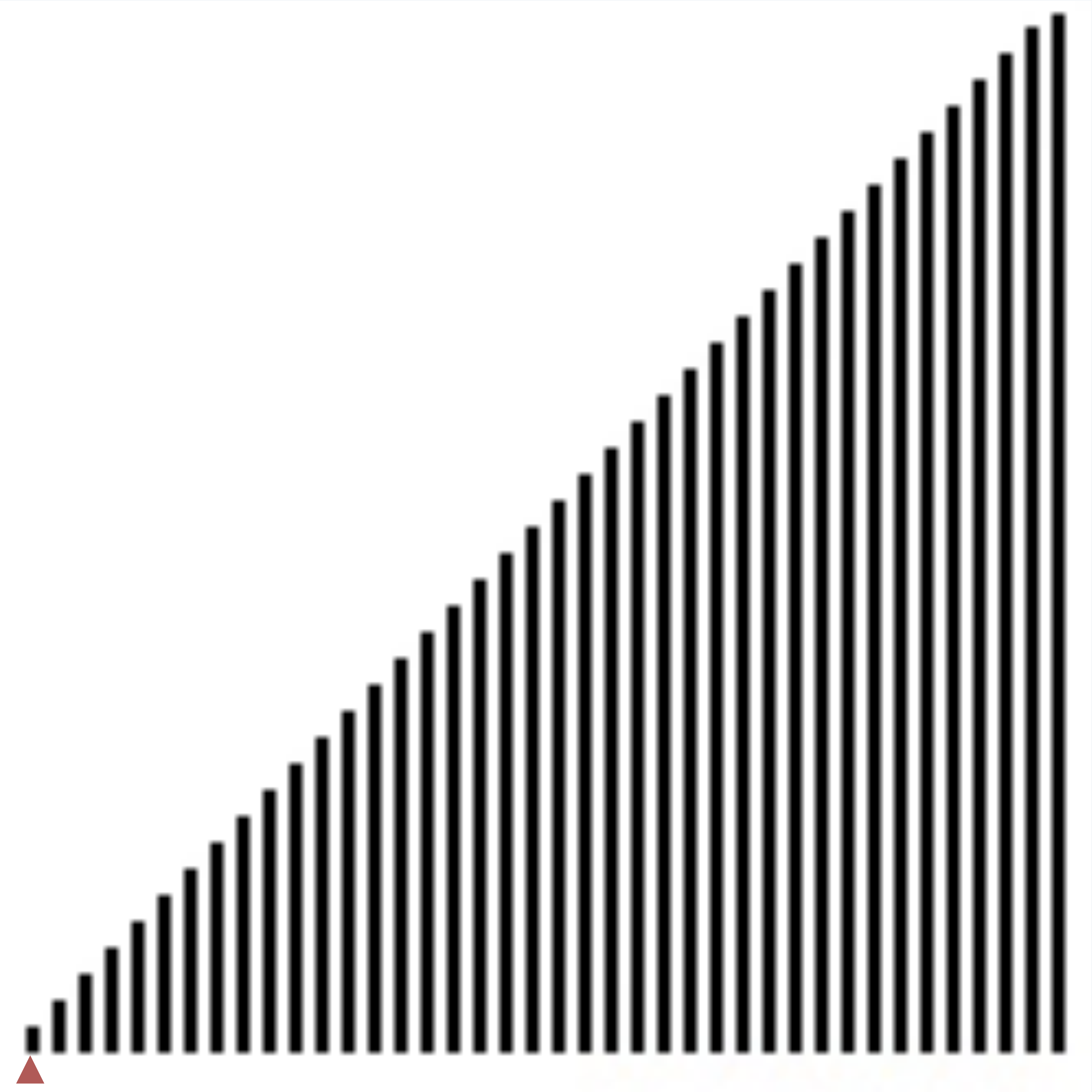
$0 + 1 + 2 + \ldots + (n-1) \sim \frac{1}{2} n^2$



https://www.toptal.com/developers/sorting-algorithms/insertion-sort

Insertion sort: running time analysis

Best case. Insertion sort makes $n-1$ compares and 0 exchanges to sort an array of n distinct keys in ascending order.



algorithm position

in order

not yet seen

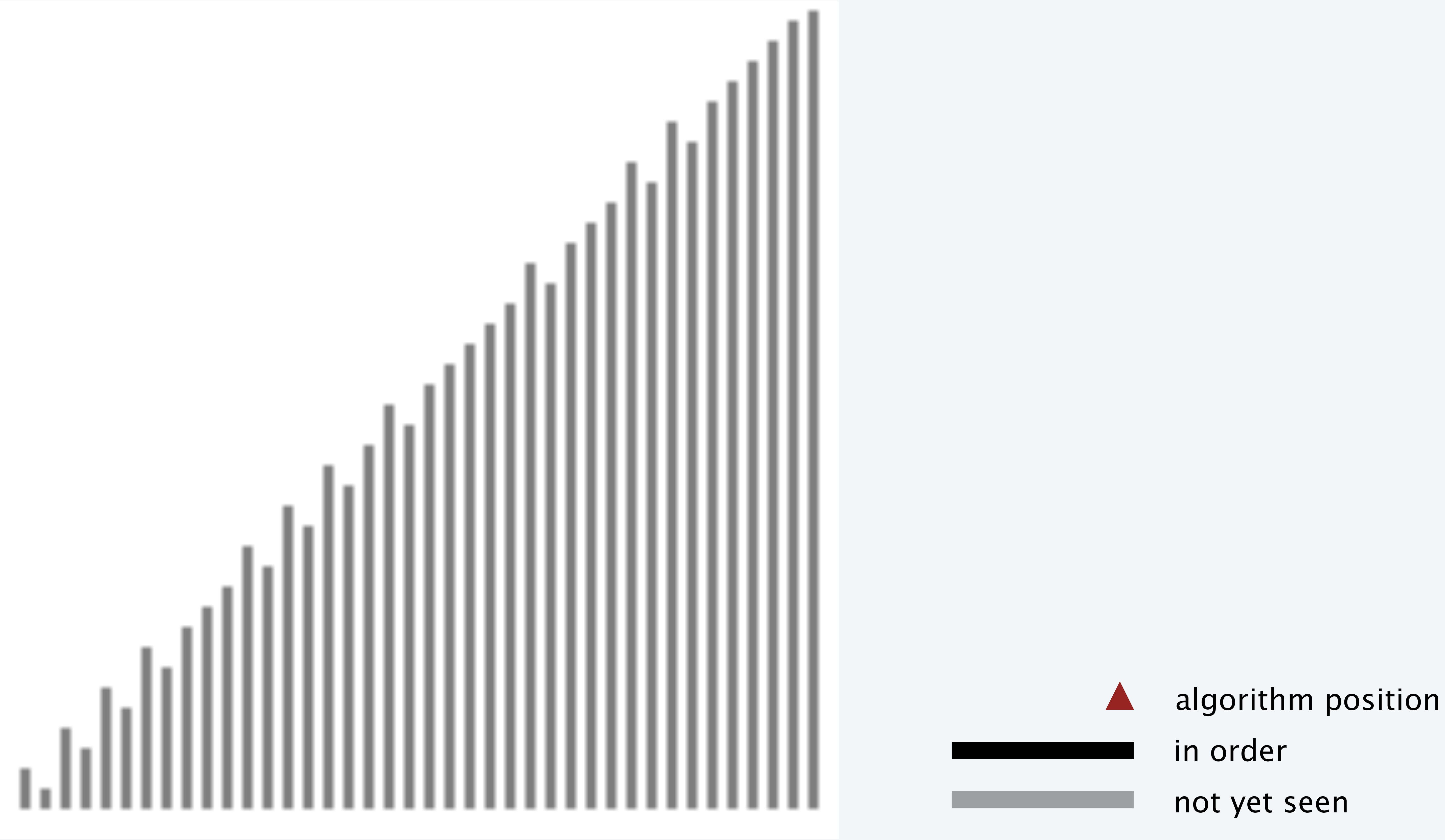
https://www.toptal.com/developers/sorting-algorithms/insertion-sort

Insertion sort: running time analysis

Good case. Insertion sort takes $\Theta(n)$ time on "partially sorted" arrays.

Q. Can we formalize what we mean by partially sorted?

A. Yes, in terms of "inversions" (see textbook).



https://www.toptal.com/developers/sorting-algorithms/insertion-sort

Insertion sort: practical improvements

Half exchanges. Shift items over (instead of exchanging).

- Same compares; fewer array accesses.
- No longer uses only `less()` and `exch()` to access data.

A C H H I M N P Q X Y K B I N A R Y

Binary insertion sort. Use binary search to find insertion point.

- Now, worst-case number of compares $\sim n \log_2 n$.
- But still makes $\Theta(n^2)$ array accesses in worst case.

A C H H I M N P Q X Y K B I N A R Y

binary search for first key > K

https://algs4.cs.princeton.edu

1.4 Analysis of Algorithms

- rules of the game
- selection sort
- insertion sort
- binary search

Binary search

Goal. Given a sorted array and a search key, find index of the search key in the array?

Binary search. Compare search key with middle entry.

- Too small, go left.
- Too big, go right.
- Equal, found.

sorted array

6	13	14	25	33	43	51	53	64	72	84	93	95	96	97
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
↑ lo														↑ hi

Binary search: implementation

Trivial to implement?

- First binary search published in 1946.
- First bug-free one in 1962.
- Bentley experiment: 90% of programmers implement it incorrectly.
- Bug in Java's `Arrays.binarySearch()` discovered in 2006.

and in C, C++, …

Extra, Extra - Read All About It: Nearly All Binary Searches and Mergesorts are Broken

Friday, June 02, 2006

Posted by Joshua Bloch, Software Engineer

I remember vividly Jon Bentley's first Algorithms lecture at CMU, where he asked all of us incoming Ph.D. students to write a binary search, and then dissected one of our implementations in front of the class. Of course it was broken, as were most of our implementations. This made a real impression on me, as did the treatment of this material in his wonderful *Programming Pearls* (Addison-Wesley, 1986; Second Edition, 2000). The key lesson was to carefully consider the invariants in your programs.

https://ai.googleblog.com/2006/06/extra-extra-read-all-about-it-nearly.html

Binary search: implementation

Invariant. If `key` appears in array `a[]`, then `a[lo]` ≤ `key` ≤ `a[hi]`.

```
public static int binarySearch(String[] a, String key) {
   int lo = 0, hi = a.length - 1;
   while (lo <= hi) {
      int mid = (lo + hi) >>> 1;
      int compare = key.compareTo(a[mid]);
      if      (compare < 0) hi = mid - 1;
      else if (compare > 0) lo = mid + 1;
      else return mid;
   }
   return -1;
}
```

why not `mid = (lo + hi) / 2` ?

https://algs4.cs.princeton.edu/11model/BinarySearch.java.html

Binary search: analysis

Proposition. Binary search makes at most $1 + \log_2 n$ compares to search in any sorted array of length n.

Pf.

- Each iteration of `while` loop:
  - calls `compareTo()` once
  - decreases the length of remaining subarray by at least a factor of 2

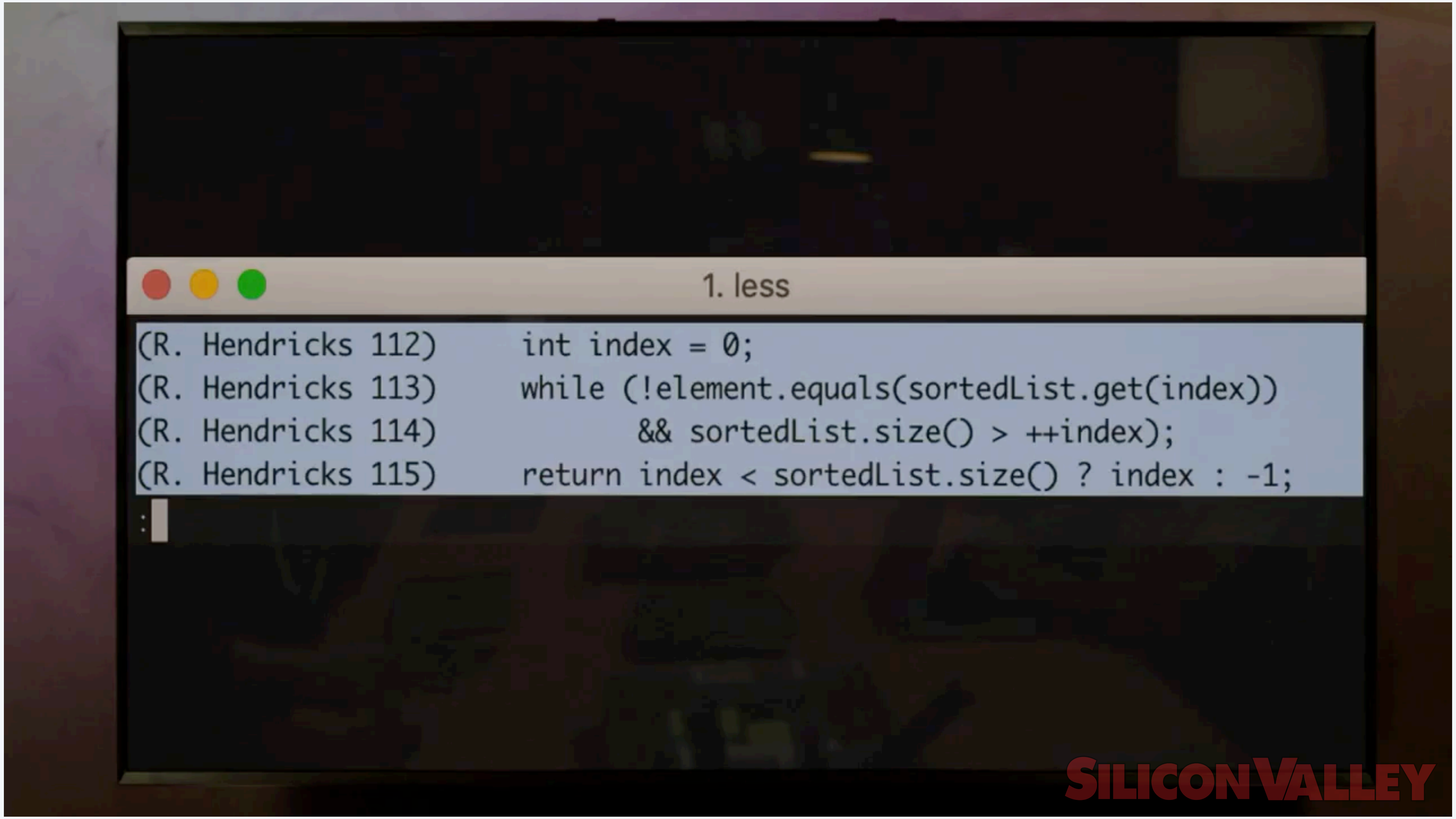
can happen at most $1 + \log_2 n$ times. Why?

$$n \rightarrow n/2 \rightarrow n/4 \rightarrow n/8 \rightarrow \cdots \rightarrow 2 \rightarrow 1$$

$$\underbrace{\qquad\qquad}_{1 + \log_2 n}$$

slightly better than 2×, due to elimination of `a[mid]` *from subarray (or early termination of* `while` *loop)*

Binary search vs. sequential search



3-SUM. Given an array of n distinct integers, count number of triples that sum to 0.

Version 0. $\Theta(n^3)$ time in worst case. ✔

Version 1. $\Theta(n^2 \log n)$ time in worst case.

Version 2. $\Theta(n^2)$ time in worst case.

Note. For full credit, use only $\Theta(1)$ extra space.

Summary

Comparable interface. Java framework for comparing items.

Selection sort. $\Theta(n^2)$ compares; $\Theta(n)$ exchanges.

Insertion sort. $\Theta(n^2)$ compares and exchanges in the worst case.

Binary search. Search a sorted array using $\Theta(\log n)$ compares in worst case.

Credits

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Anime Boy	freesvg.org	public domain
Anime Girl	freesvg.org	public domain
Balance	Adobe Stock	education license
Jon Bentley	Amazon	
Binary vs. Sequential Search	Silicon Valley S6E4	
Insertion Sort Dance	AlgoRythmics	

Insertion sort with Romanian folk dance

https://www.youtube.com/watch?v=ROalU379l3U