Parametric Surfaces

COS 426, Fall 2022

PRINCETON UNIVERSITY

3D Object Representations

- Points
  - Range image
  - Point cloud
- Surfaces
  - Polygonal mesh
  - ➢ Parametric
  - Subdivision
  - Implicit
- Solids
  - Voxels
  - BSP tree
  - CSG
  - Sweep
- High-level structures
  - Scene graph
  - Application specific

Parametric Surfaces

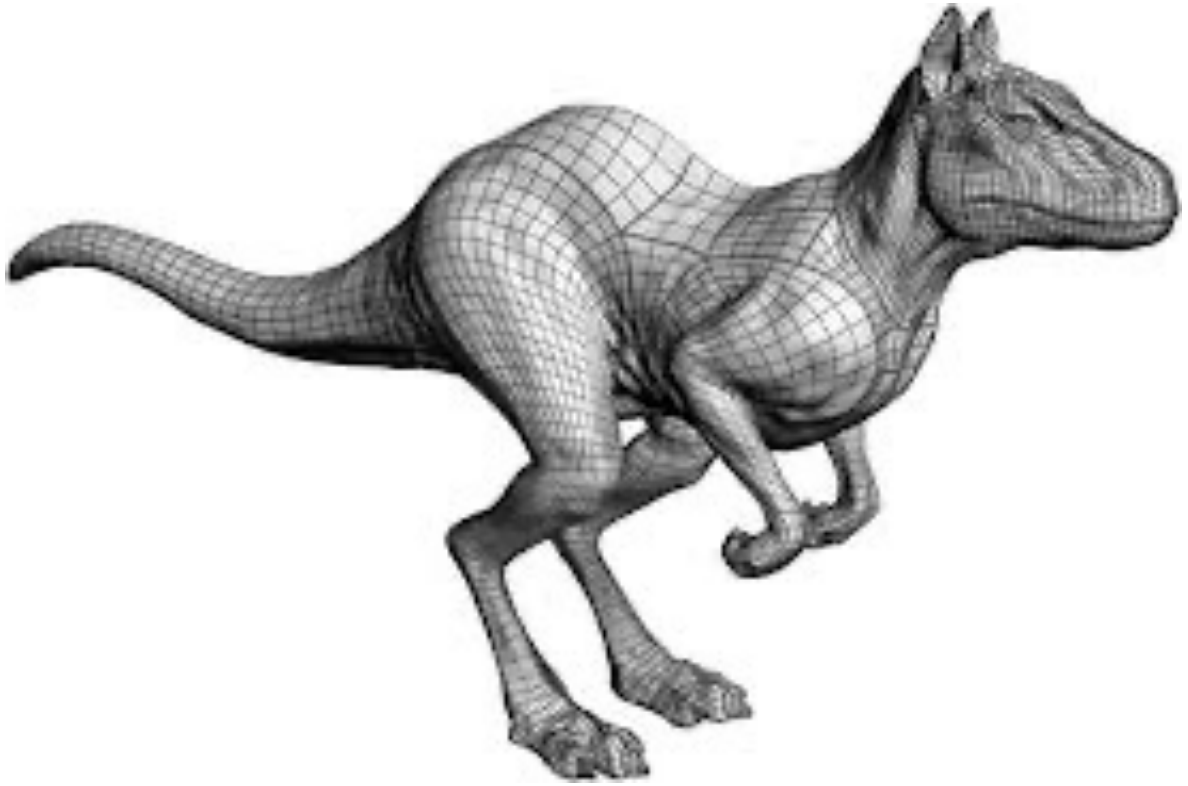
- Applications
  - Design of smooth surfaces in cars, airplanes, ships, etc.

Parametric Surfaces

- Applications
  - Partitioning surfaces into patches

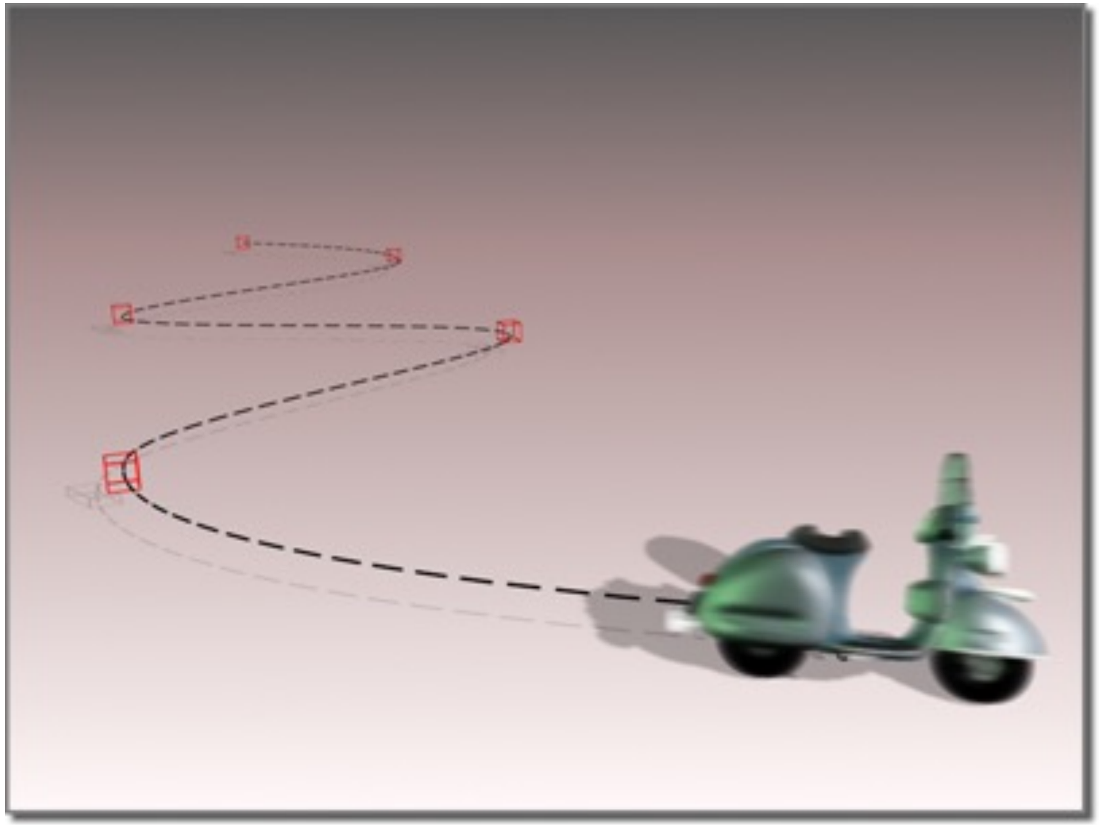
Parametric Surfaces

- Applications
  - Creating characters or scenes for movies



Parametric Curves

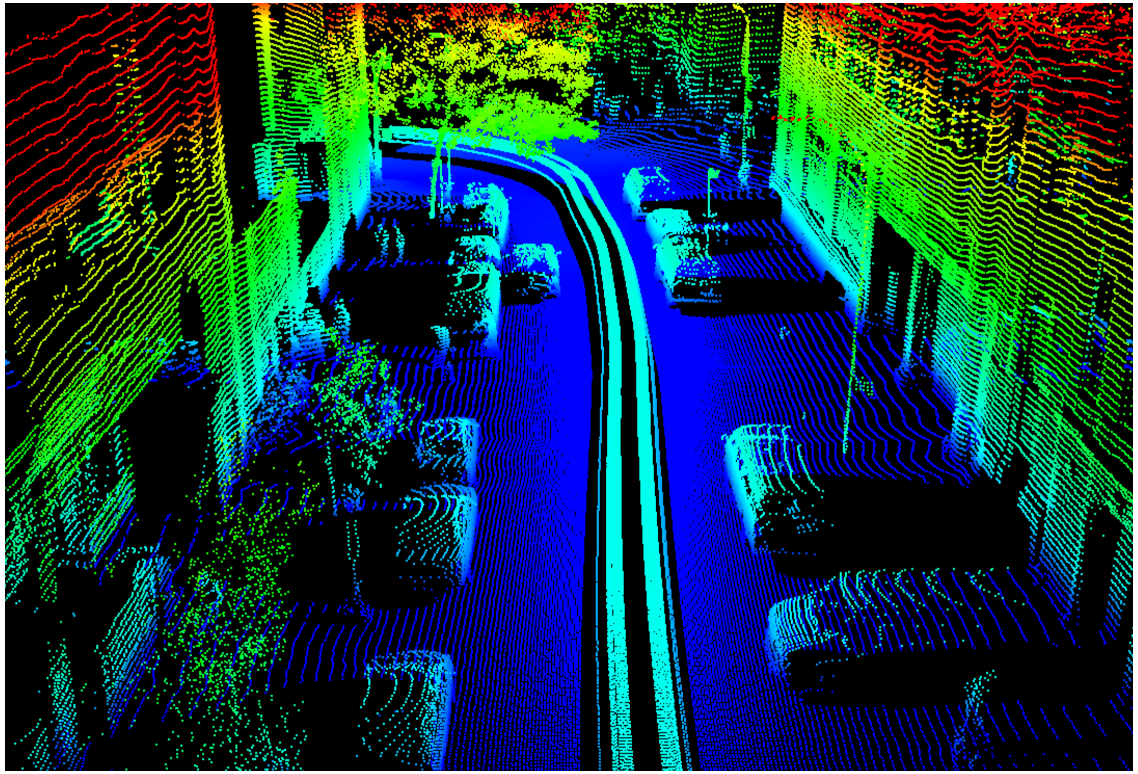
- Applications
  - Defining motion trajectories for objects or cameras



AutoDesk

Parametric Curves

- Applications
  - Defining smooth interpolations of sparse data



Google Street View

Outline

- Parametric curves
  - Cubic B-Spline
  - Cubic Bézier
- Parametric surfaces
  - Bi-cubic B-Spline
  - Bi-cubic Bézier

Outline

- Parametric curves
  - Cubic B-Spline
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- Parametric surfaces
  - Bi-cubic B-Spline
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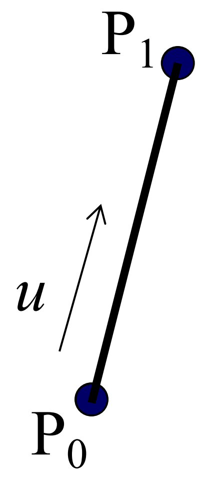
Parametric Curves

- Defined by parametric functions:
  - $x = f_x(u)$
  - $y = f_y(u)$

- Example: line segment

$$f_x(u) = (1-u)x_0 + ux_1$$
$$f_y(u) = (1-u)y_0 + uy_1$$
$$u \in [0..1]$$



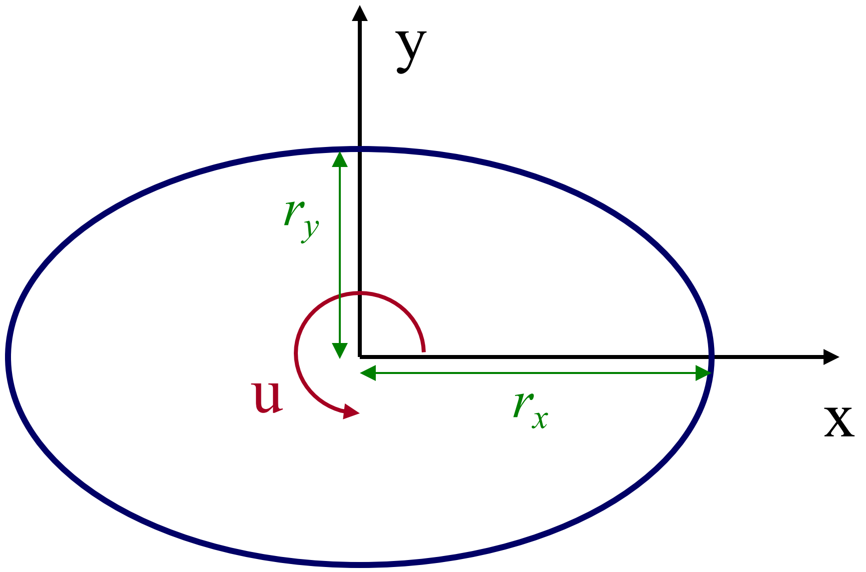
H&B Figure 10.10

Parametric Curves

- Defined by parametric functions:
  - $x = f_x(u)$
  - $y = f_y(u)$

- Example: ellipse

$$f_x(u) = r_x \cos(2\pi u)$$
$$f_y(u) = r_y \sin(2\pi u)$$
$$u \in [0..1]$$

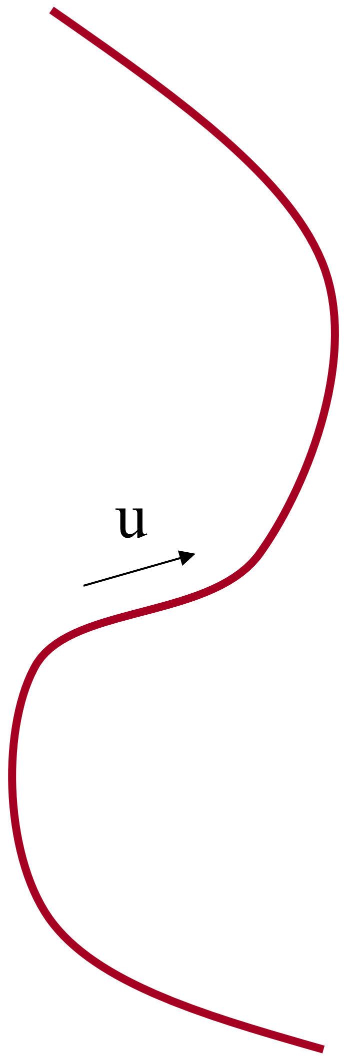


H&B Figure 10.10

Parametric curves

- How to easily define arbitrary curves?

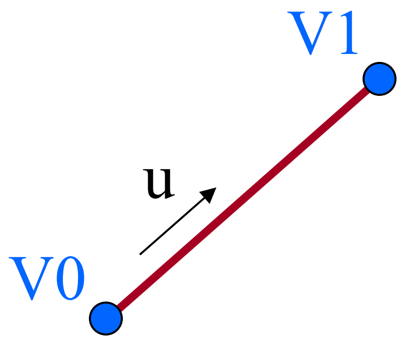
$$x = f_x(u)$$
$$y = f_y(u)$$



Parametric curves

- How to easily define arbitrary curves?

$$x = f_x(u)$$
$$y = f_y(u)$$



- Use functions that “blend” user-defined *control points*

$$x = f_x(u) = V0_x*(1 - u) + V1_x*u$$
$$y = f_y(u) = V0_y*(1 - u) + V1_y*u$$

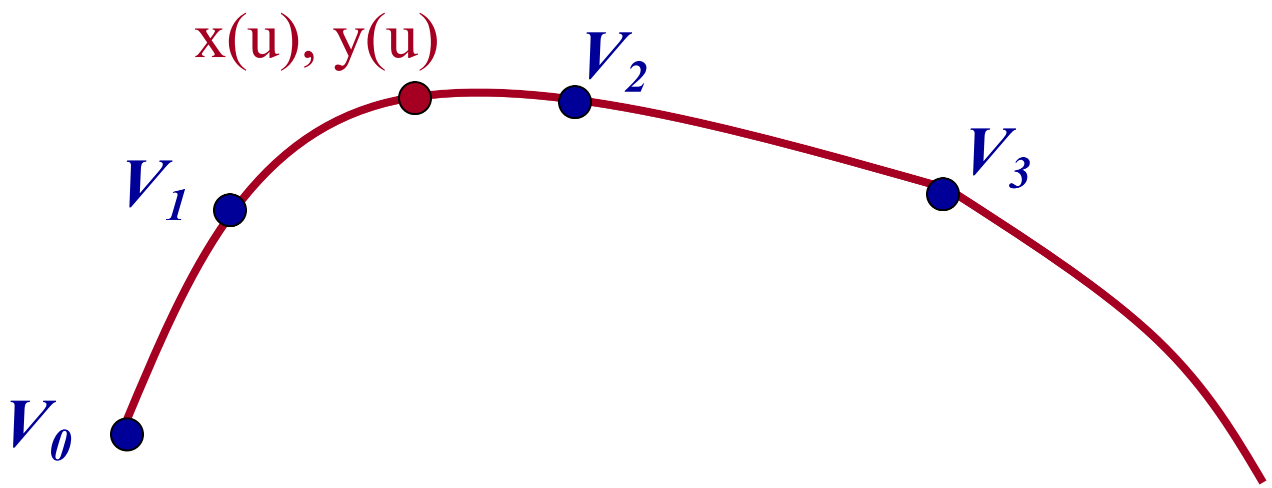
Simple functions of u

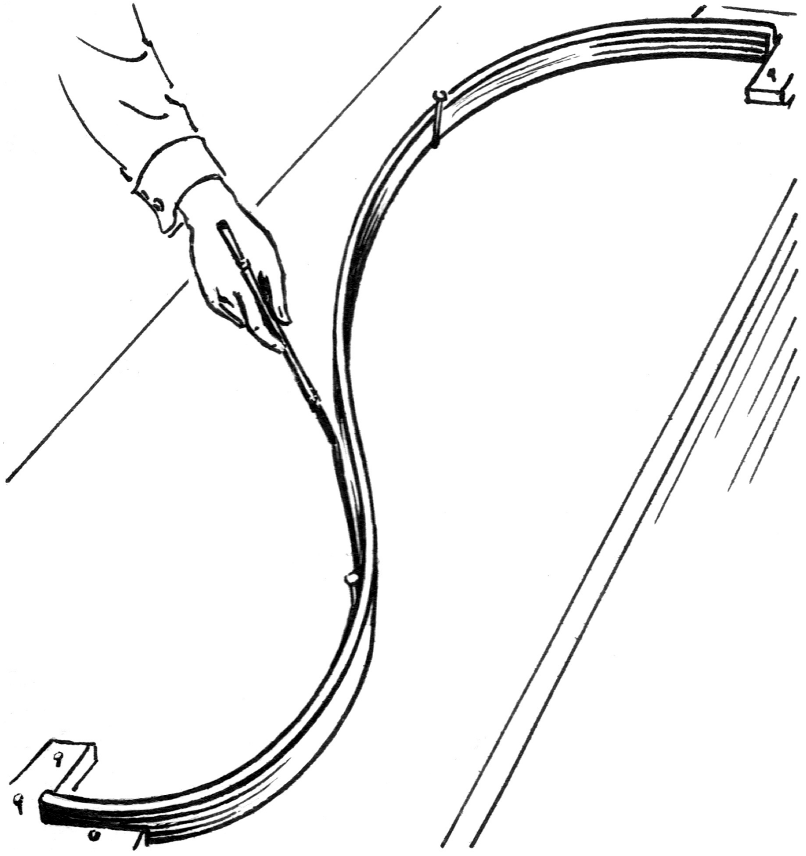
Parametric curves

- More generally:

$$x(u) = \sum_{i=0}^{n} B_i(u) * Vi_x$$

$$y(u) = \sum_{i=0}^{n} B_i(u) * Vi_y$$





[Wikipedia]

Parametric curves

- What B(u) functions should we use?

$$x(u) = \sum_{i=0}^{n} B_i(u) * Vi_x$$

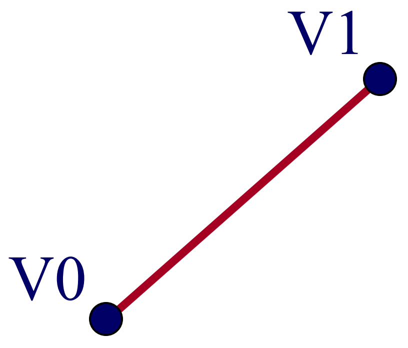
$$y(u) = \sum_{i=0}^{n} B_i(u) * Vi_y$$

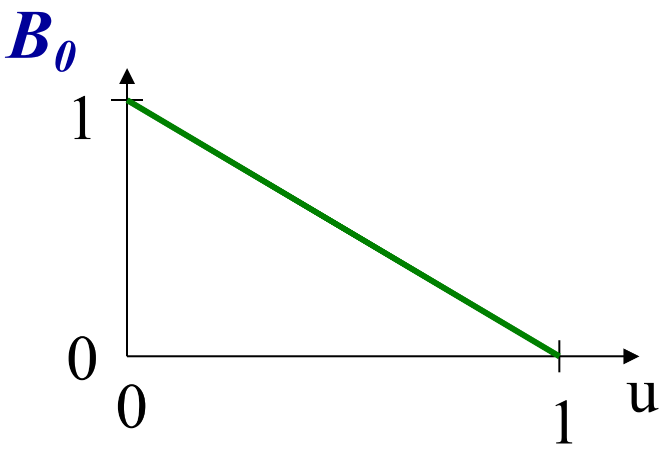
Parametric curves

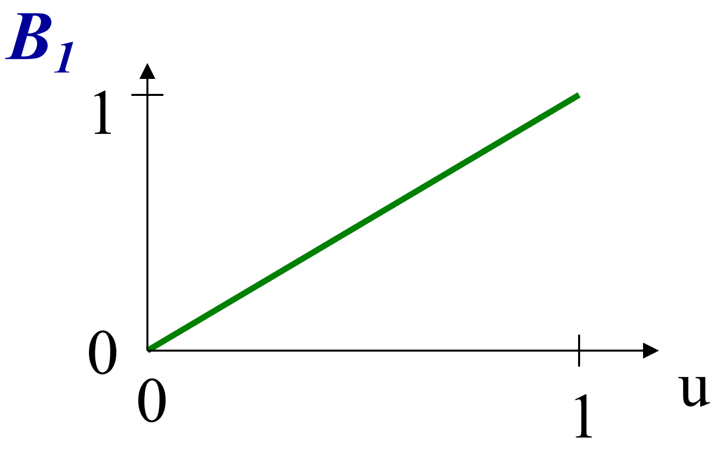
- What B(u) functions should we use?

$$x(u) = \sum_{i=0}^{n} B_i(u) * Vi_x$$

$$y(u) = \sum_{i=0}^{n} B_i(u) * Vi_y$$





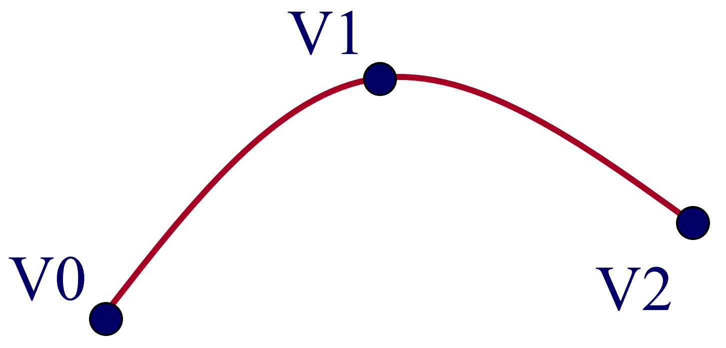


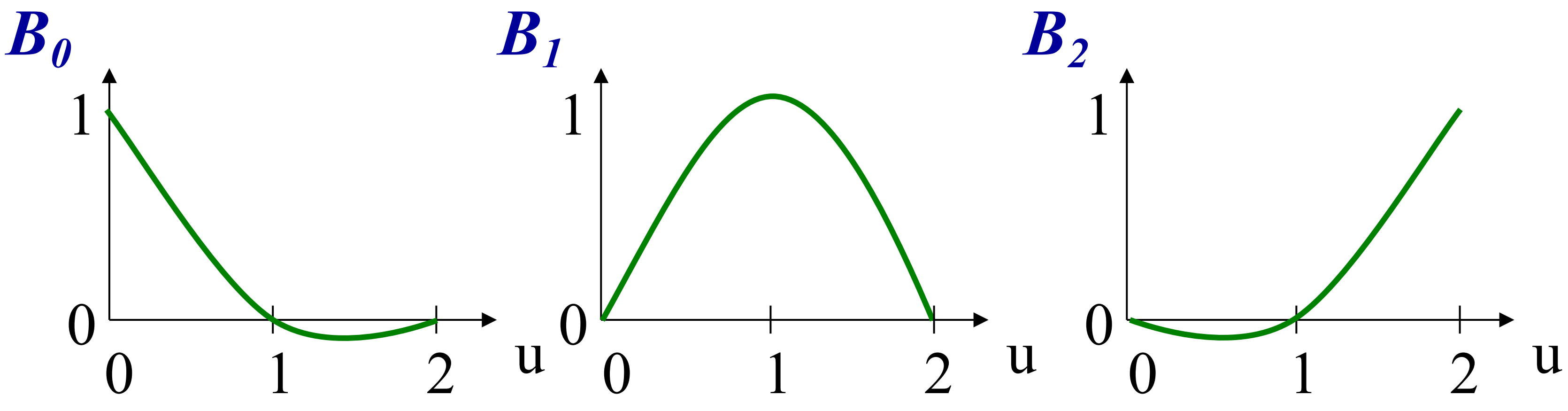
Parametric curves

- What B(u) functions should we use?

$$x(u) = \sum_{i=0}^{n} B_i(u) * Vi_x$$

$$y(u) = \sum_{i=0}^{n} B_i(u) * Vi_y$$

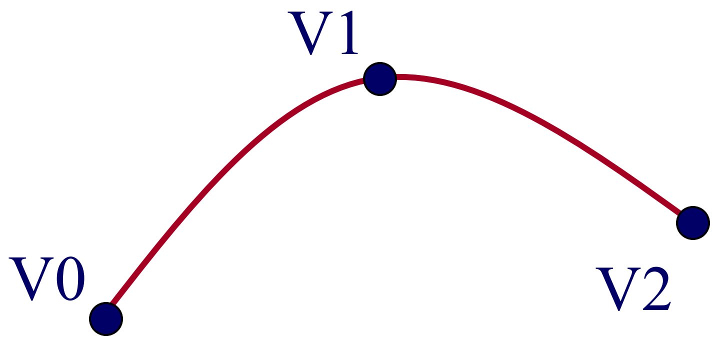




Parametric Polynomial Curves

- Polynomial blending functions:

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$

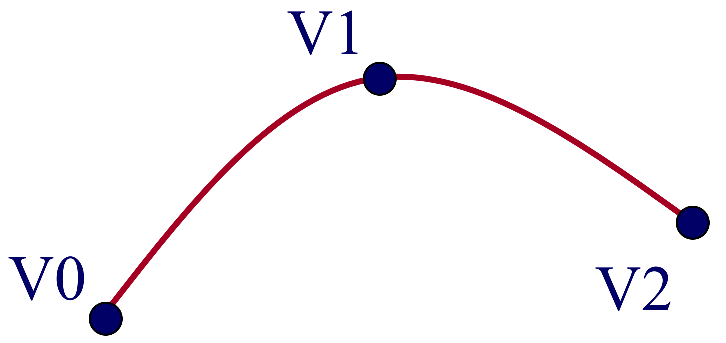


- Advantages of polynomials
  - Easy to compute
  - Infinitely differentiable
  - Easy to derive curve properties

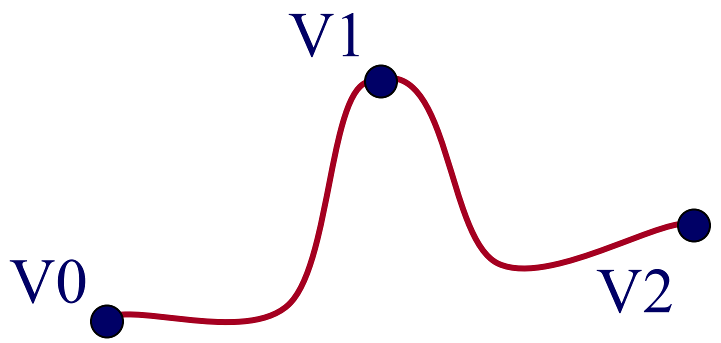
Parametric Polynomial Curves

- Polynomial blending functions:

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$

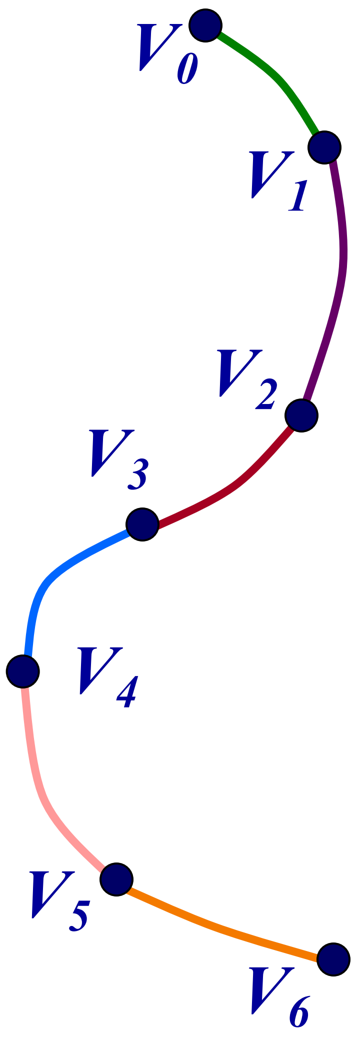


- What degree polynomial?
  - For expressivity:
    high-degree polynomials preferred
  - For ease of computation and control:
    low-degree polynomials preferred



Piecewise Parametric Polynomial Curves

- Splines:
  - Split curve into segments
  - Each segment defined by *low-order* polynomial blending *subset* of control vertices
- Motivation:
  - Same blending functions for every segment
  - Prove properties from blending functions
  - Provides local control & efficiency
- Challenges
  - How to choose blending functions?
  - How to determine properties?



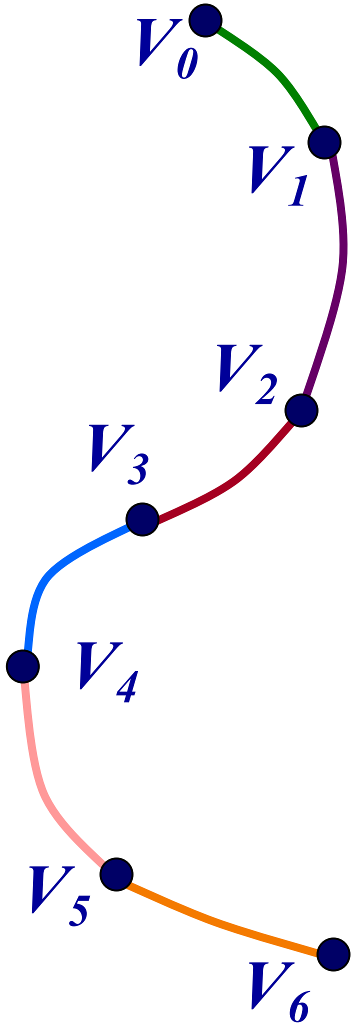
Splines

- Some properties we might like to have:
  - Local control
  - Continuity
  - Interpolation?
  - Convex hull?

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$

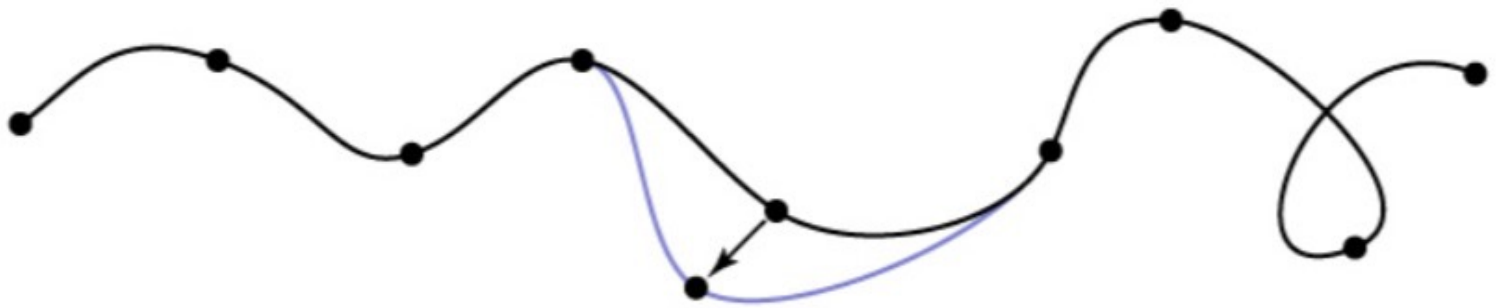
Blending functions determine properties

Properties determine blending functions

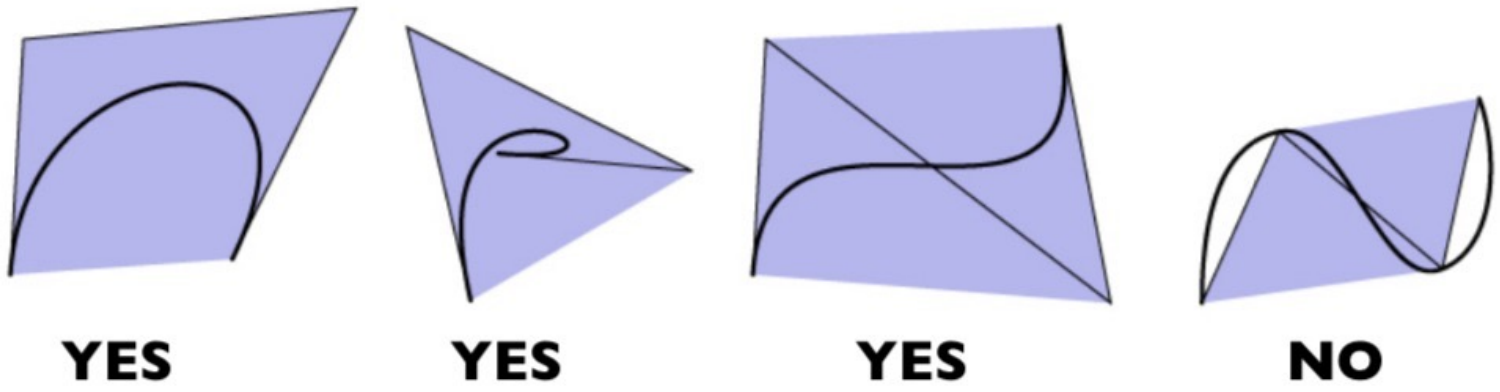


Splines

- Local control:



- Convex hull property:



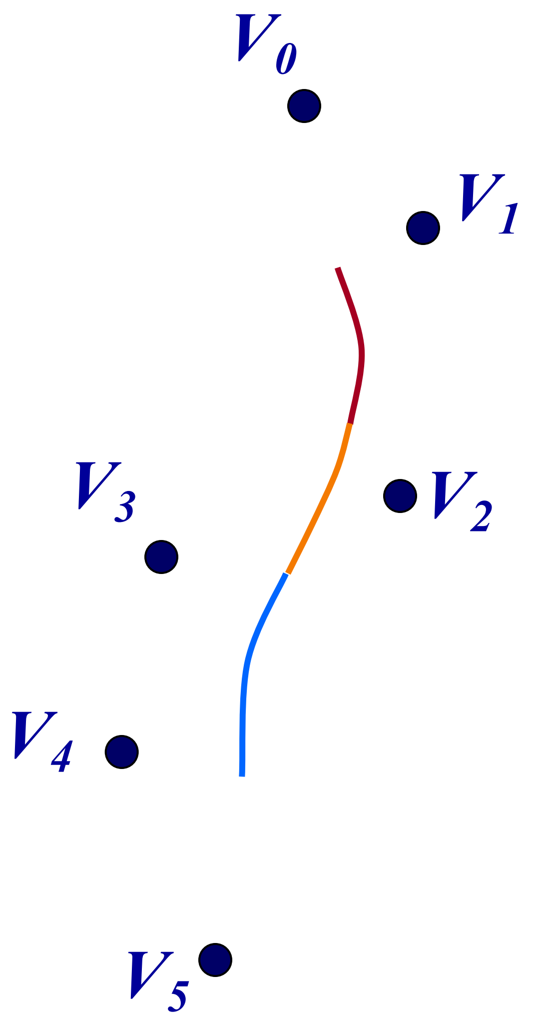
[Marschner]

Outline

- Parametric curves
  - Cubic B-Spline
  - Cubic Bézier
- Parametric surfaces
  - Bi-cubic B-Spline
  - Bi-cubic Bézier

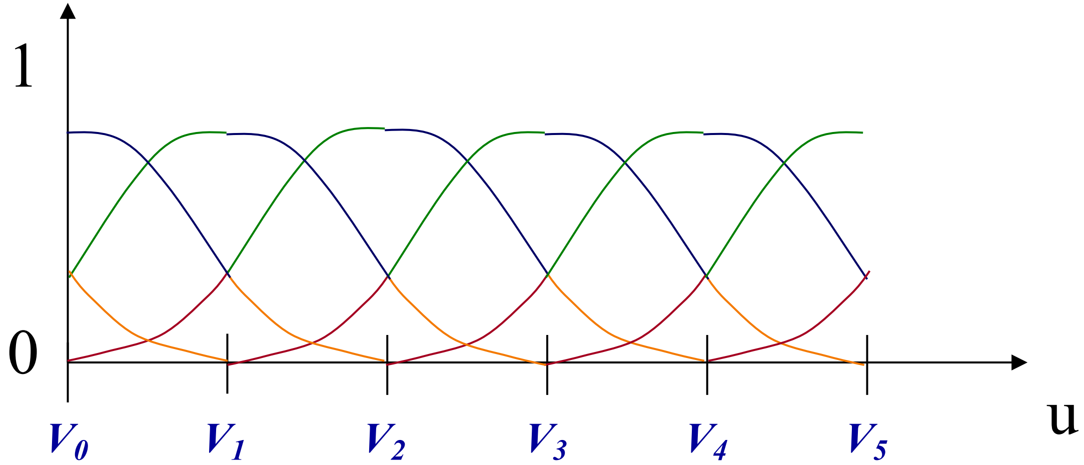
Cubic B-Splines

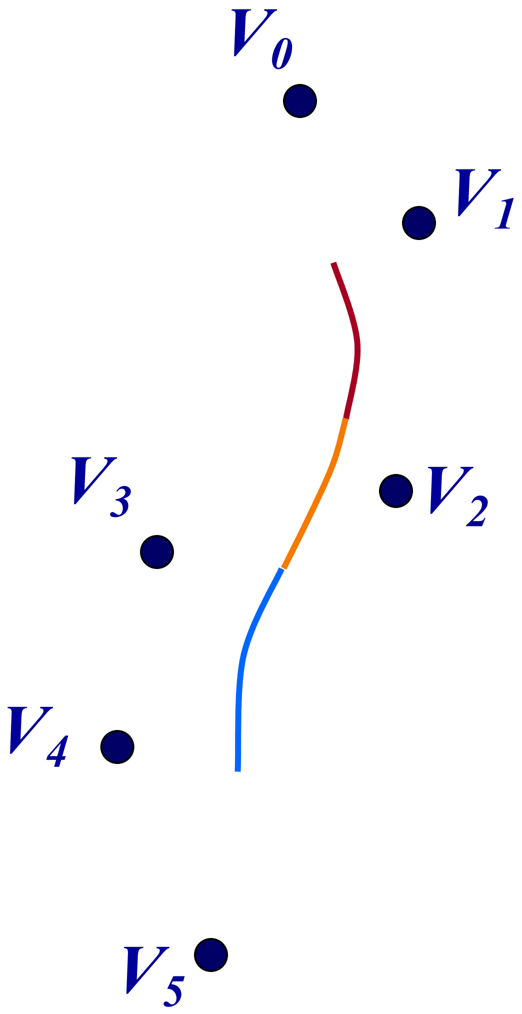
- Properties:
  - Local control
  - C^2 continuity at joints
    (infinitely continuous within each piece)
  - Approximating
  - Convex hull



Cubic B-Spline Blending Functions

- Blending functions imply properties:
  - Local control
  - C^2 continuity at joints (infinitely continuous within each piece)
  - Approximating
  - Convex hull

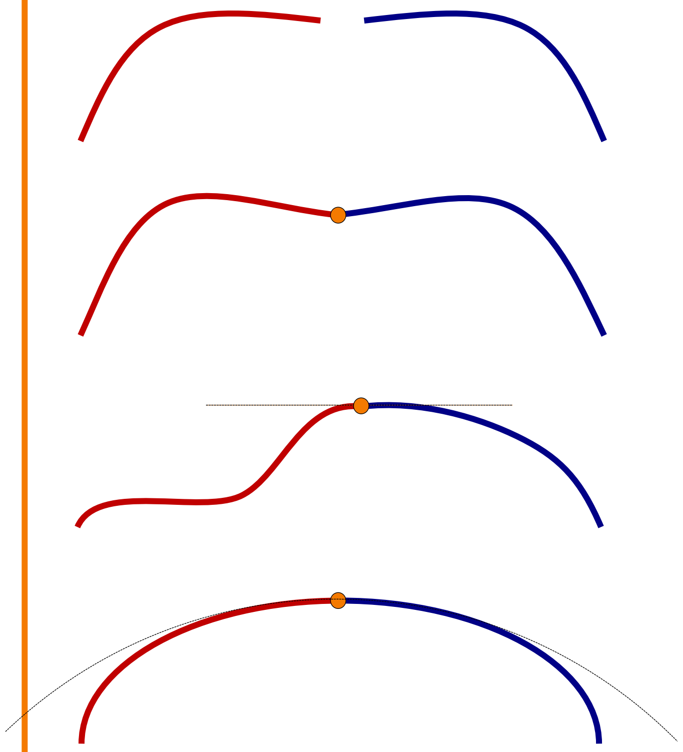




http://www.ibiblio.org/e-notes/Splines/basis.html

Cubic B-Splines

- Defining continuity at joints
  - Geometric continuity:



“G^{-1}”: discontinuous

G^0: touching

G^1: same tangent

G^2: same curvature

  - Parametric continuity:
    Also considers derivative of the parametric function(s) along path

C^0: same as G^0

C^1: same velocity

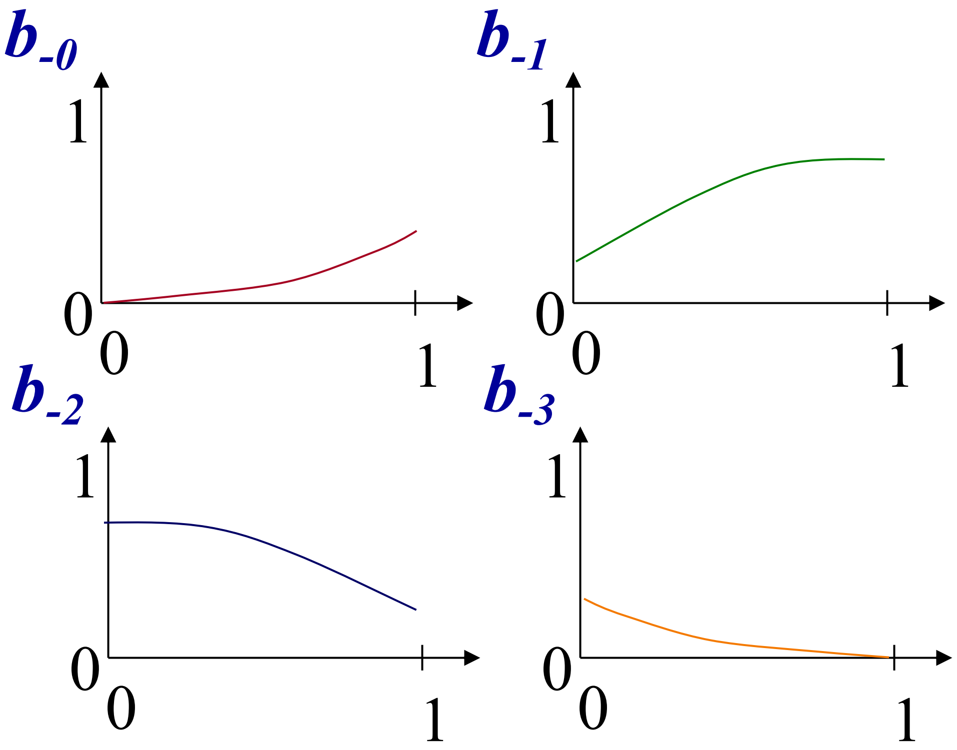
C^2: same acceleration

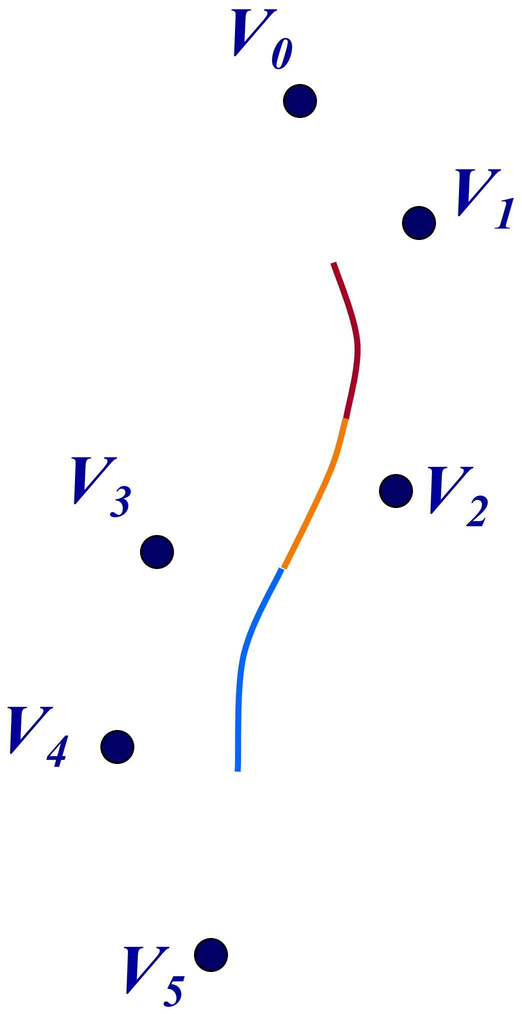
Each G^n implies all lower ones, each C^n implies all lower ones

Cubic B-Spline Blending Functions

- Blending functions:

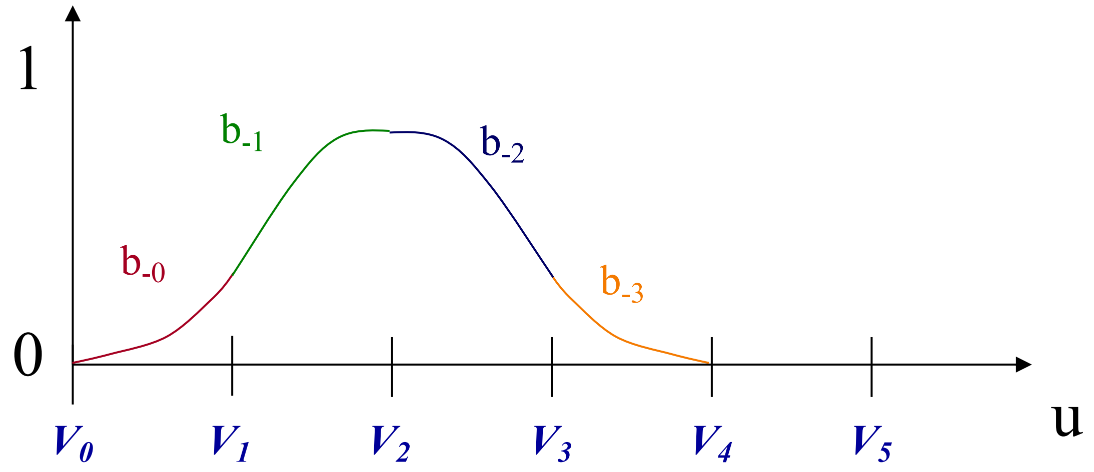
$$B_i(u)=\sum_{j=0}^{m} a_j u^j$$

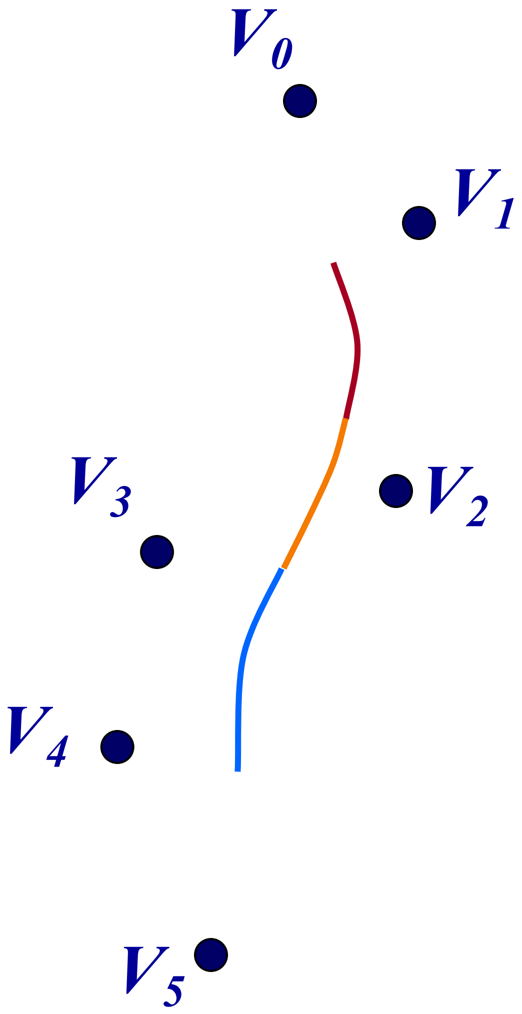




Cubic B-Spline Blending Functions

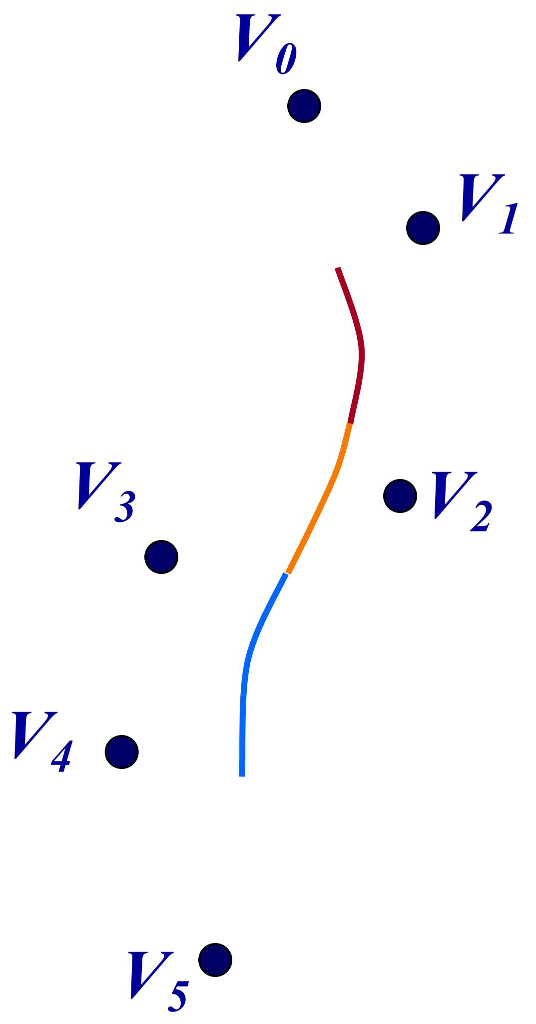
- How derive blending functions?
  - Cubic polynomials
  - Local control
  - C^2 continuity
  - Convex hull





Cubic B-Spline Blending Functions

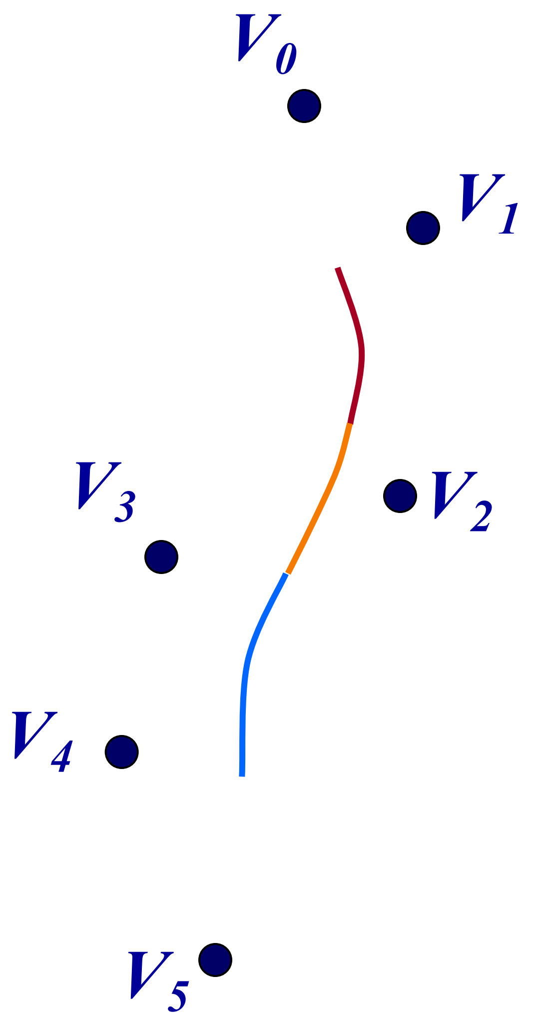
- Four cubic polynomials for four vertices
  - 16 variables (degrees of freedom)
  - Variables are a_i, b_i, c_i, d_i for four blending functions



$$
\begin{aligned}
b_{-0}(u) &= a_0 u^3 + b_0 u^2 + c_0 u^1 + d_0 \\
b_{-1}(u) &= a_1 u^3 + b_1 u^2 + c_1 u^1 + d_1 \\
b_{-2}(u) &= a_2 u^3 + b_2 u^2 + c_2 u^1 + d_2 \\
b_{-3}(u) &= a_3 u^3 + b_3 u^2 + c_3 u^1 + d_3
\end{aligned}
$$

Cubic B-Spline Blending Functions

- C^2 continuity implies 15 constraints
  - Position of two curves same
  - Derivative of two curves same
  - Second derivatives same



Cubic B-Spline Blending Functions

Fifteen continuity constraints:

$$
\begin{aligned}
0 &= b_{-0}(0) \\
b_{-0}(1) &= b_{-1}(0) \\
b_{-1}(1) &= b_{-2}(0) \\
b_{-2}(1) &= b_{-3}(0) \\
b_{-3}(1) &= 0
\end{aligned}
\qquad
\begin{aligned}
0 &= b_{-0}'(0) \\
b_{-0}'(1) &= b_{-1}'(0) \\
b_{-1}'(1) &= b_{-2}'(0) \\
b_{-2}'(1) &= b_{-3}'(0) \\
b_{-3}'(1) &= 0
\end{aligned}
\qquad
\begin{aligned}
0 &= b_{-0}''(0) \\
b_{-0}''(1) &= b_{-1}''(0) \\
b_{-1}''(1) &= b_{-2}''(0) \\
b_{-2}''(1) &= b_{-3}''(0) \\
b_{-3}''(1) &= 0
\end{aligned}
$$

One more convenient constraint:

$$b_{-0}(0) + b_{-1}(0) + b_{-2}(0) + b_{-3}(0) = 1$$

Cubic B-Spline Blending Functions

- Solving the system of equations yields:

$$
\begin{aligned}
b_{-3}(u) &= -\tfrac{1}{6}u^3 + \tfrac{1}{2}u^2 - \tfrac{1}{2}u + \tfrac{1}{6} \\
b_{-2}(u) &= \tfrac{1}{2}u^3 - u^2 + \tfrac{2}{3} \\
b_{-1}(u) &= -\tfrac{1}{2}u^3 + \tfrac{1}{2}u^2 + \tfrac{1}{2}u + \tfrac{1}{6} \\
b_{-0}(u) &= \tfrac{1}{6}u^3
\end{aligned}
$$

Cubic B-Spline Blending Functions

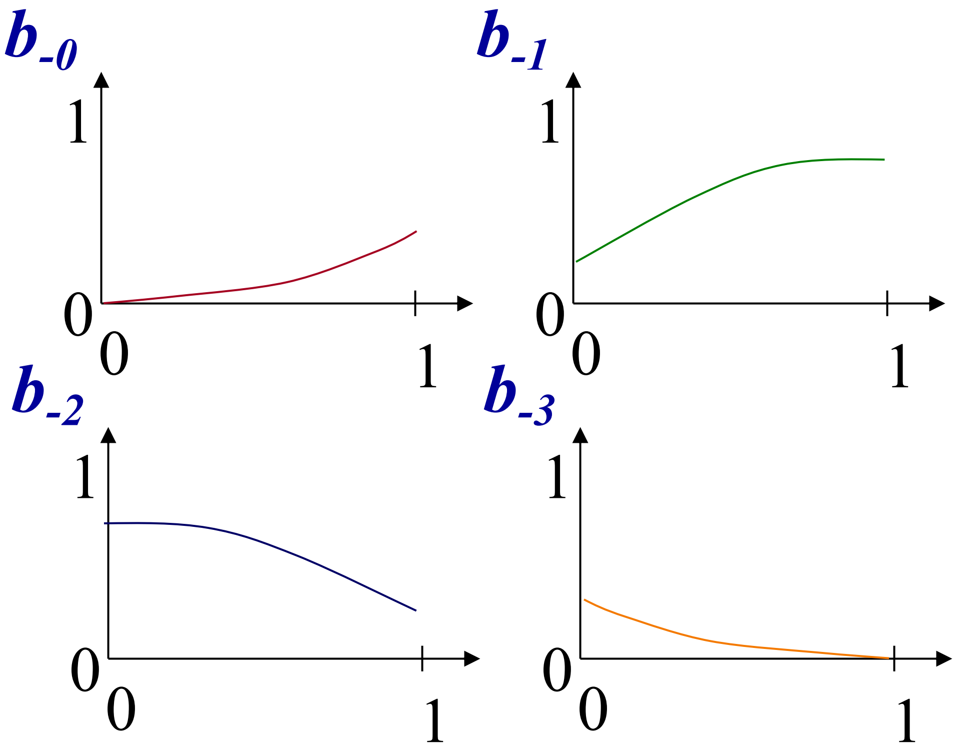
- In matrix form:

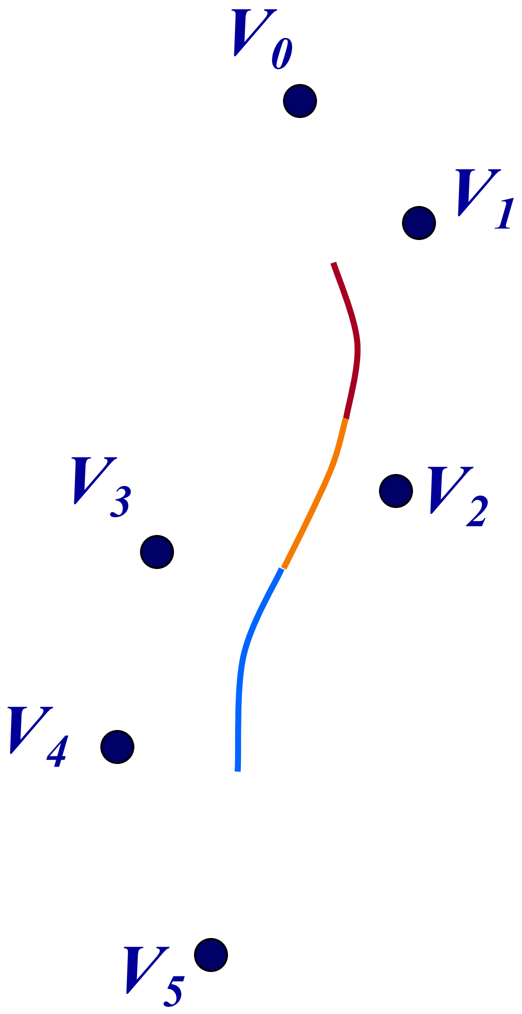
$$Q(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Cubic B-Spline Blending Functions

- In plot form:

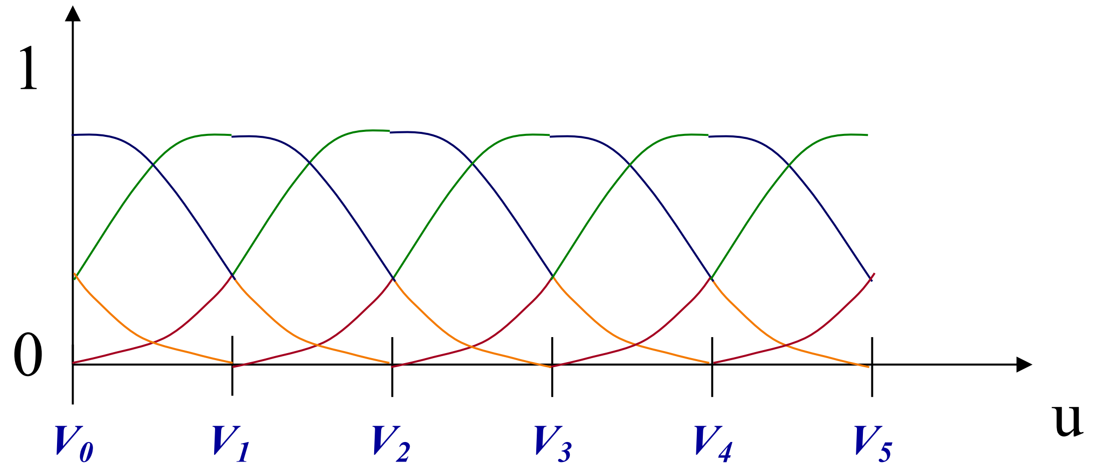
$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$

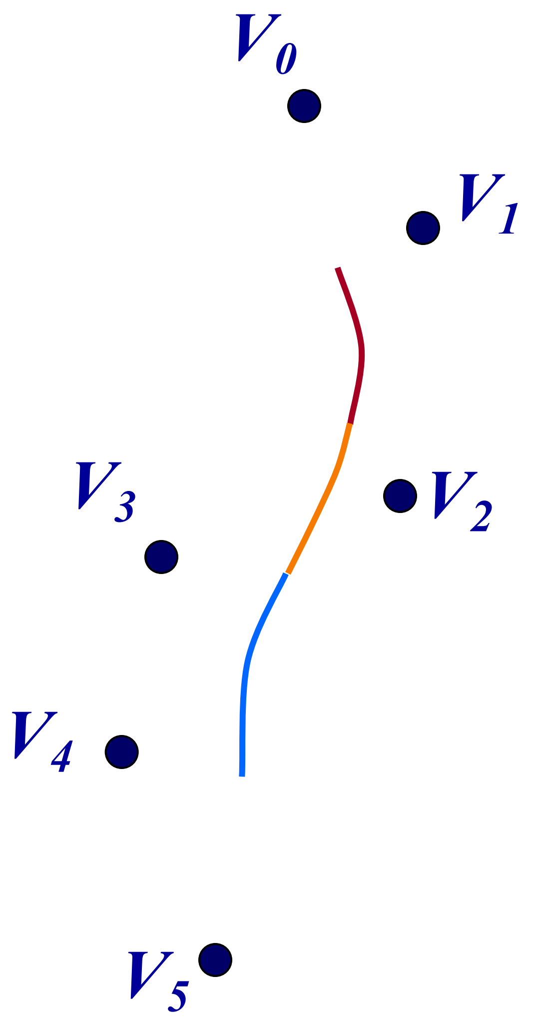




Cubic B-Spline Blending Functions

- Blending functions imply properties:
  - Local control
  - C^2 continuity at joints (infinitely continuous within each piece)
  - Approximating
  - Convex hull





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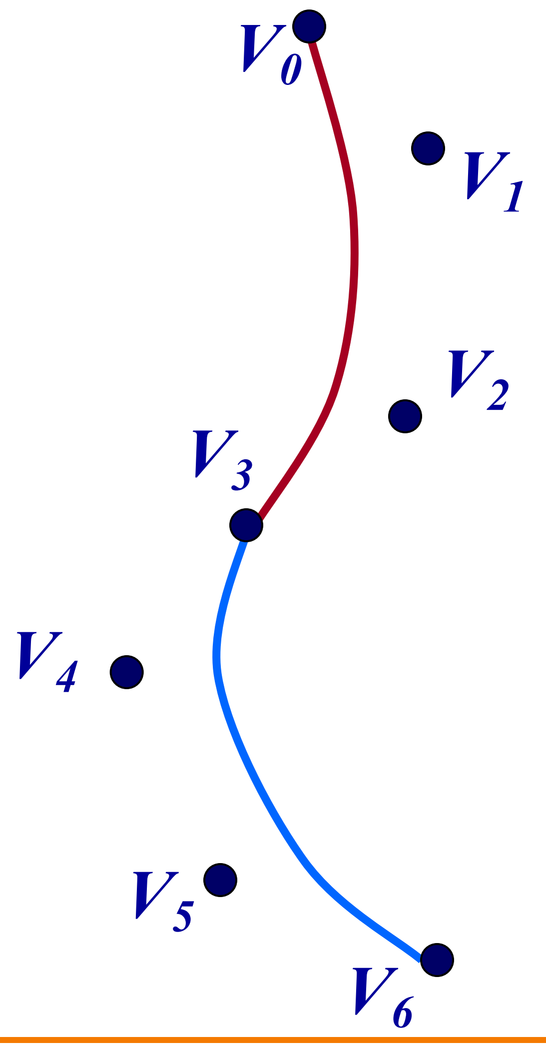
  - Cubic Bézier
- Parametric surfaces
  - Bi-cubic B-Spline
  - Bi-cubic Bézier

Bézier Curves

- Developed around 1960 by both
  - Pierre Bézier (Renault)
  - Paul de Casteljau (Citroen)
- Today: graphic design (e.g. FONTS)
- Properties:
  - Local control
  - Continuity depends on control points
  - Interpolating (every third for cubic)

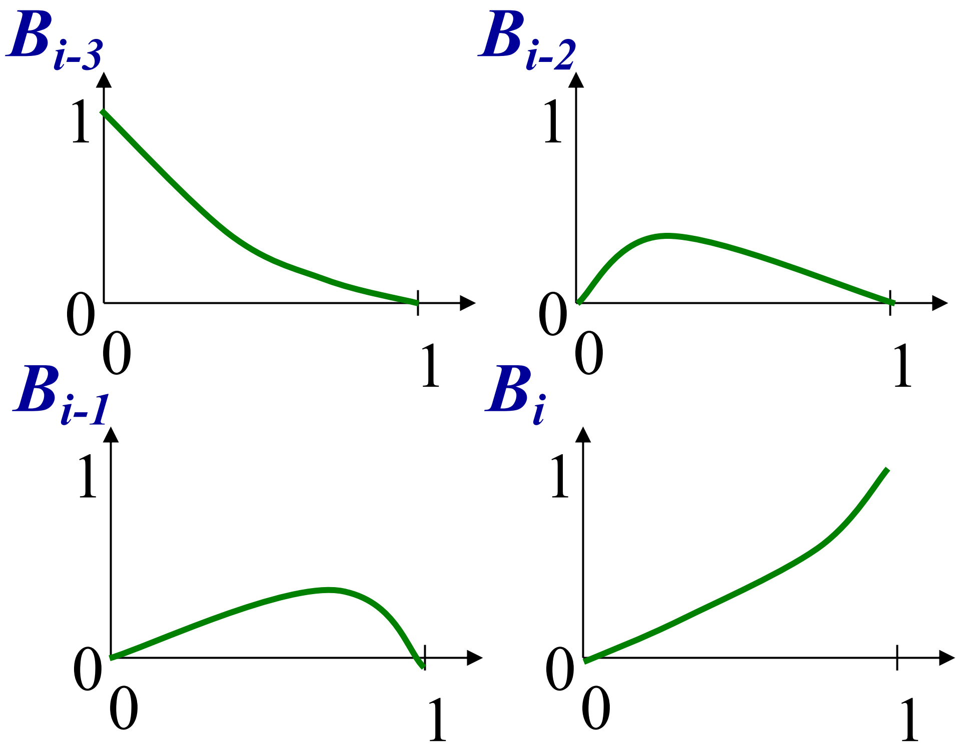
Blending functions determine properties

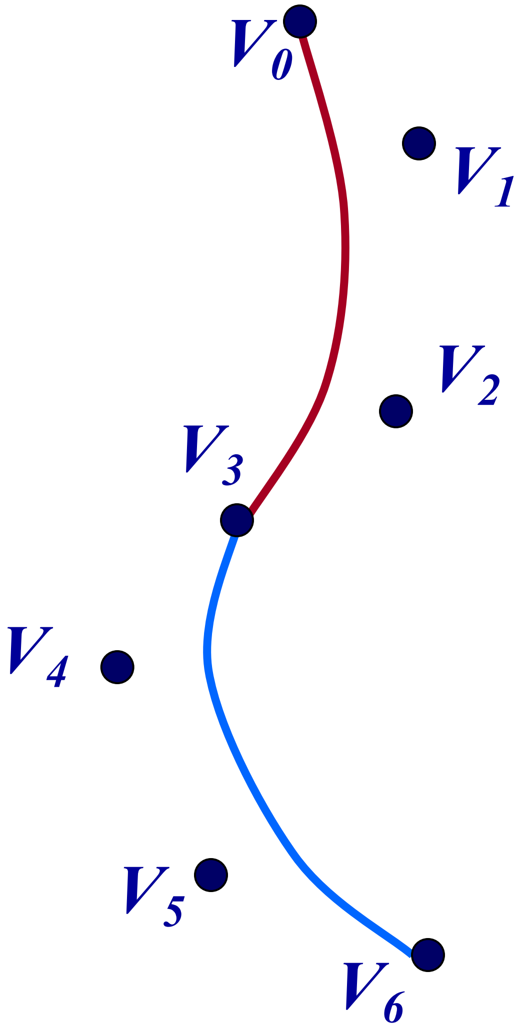


Cubic Bézier Curves

Blending functions:

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$





Cubic Bézier Curves

Bézier curves in matrix form:

$$
\begin{aligned}
Q(u) &= \sum_{i=0}^{n} V_i \binom{n}{i} u^i (1-u)^{n-i} \\
&= (1-u)^3 V_0 + 3u(1-u)^2 V_1 + 3u^2(1-u) V_2 + u^3 V_3 \\
&= \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{\mathbf{M}_{\textbf{Bézier}}} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}
\end{aligned}
$$

http://www.ibiblio.org/e-notes/Splines/bezier.html

Basic properties of Bézier Curves

- Endpoint interpolation:

$$Q(0) = V_0$$
$$Q(1) = V_n$$

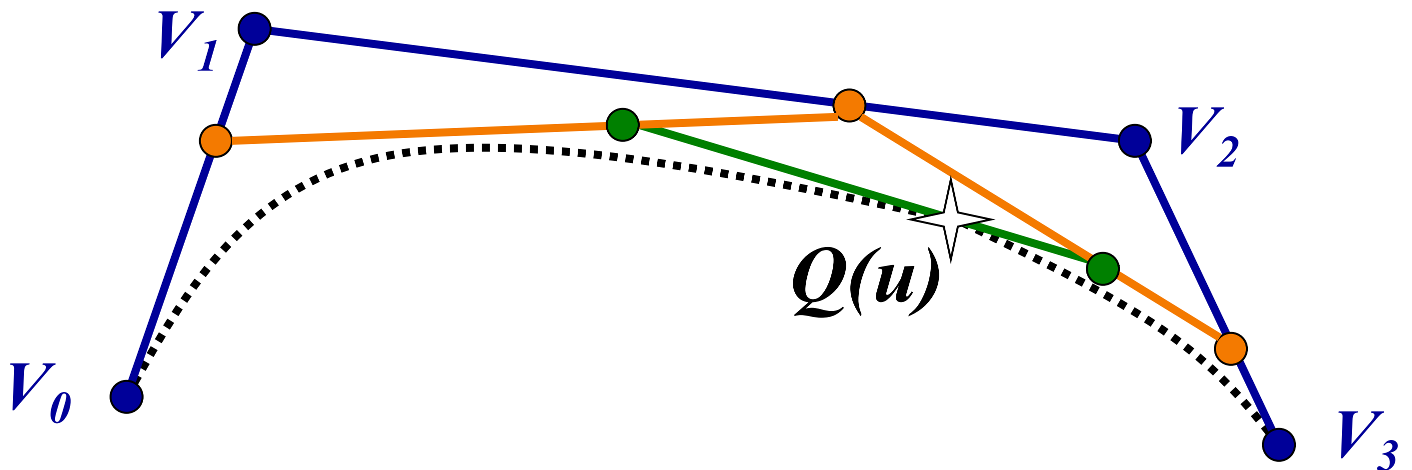
- Convex hull:
  - Curve is contained within convex hull of control polygon

- Symmetry

$$Q(u) \text{ defined by } \{V_0,\ldots,V_n\} \;\equiv\; Q(1-u) \text{ defined by } \{V_n,\ldots,V_0\}$$

Bézier Curves

- Curve $Q(u)$ can also be defined by nested interpolation:

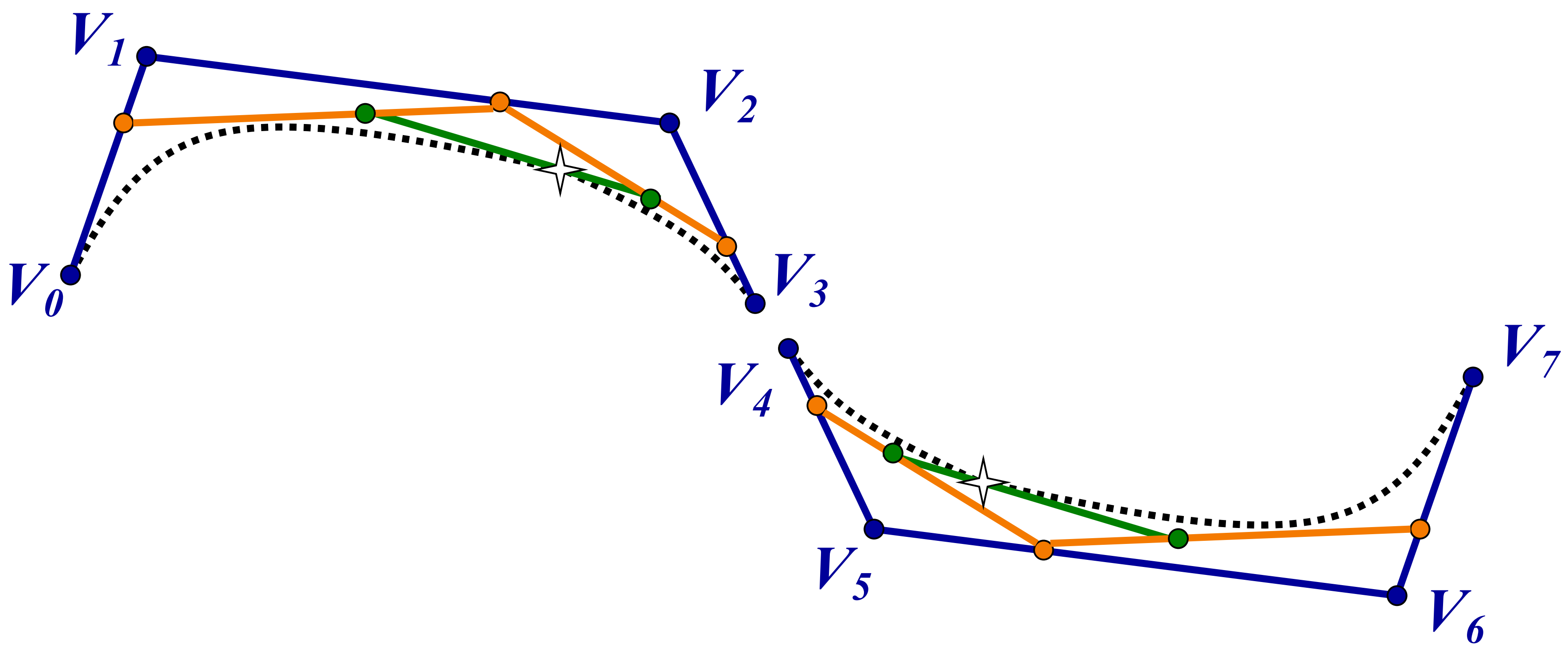


V_i are *control points*

$\{V_0, V_1, \ldots, V_n\}$ is *control polygon*

Enforcing Bézier Curve Continuity

- C^0: $V_3 = V_4$
- C^1: $V_5 - V_4 = V_3 - V_2$
- C^2: $V_6 - 2V_5 + V_4 = V_3 - 2V_2 + V_1$

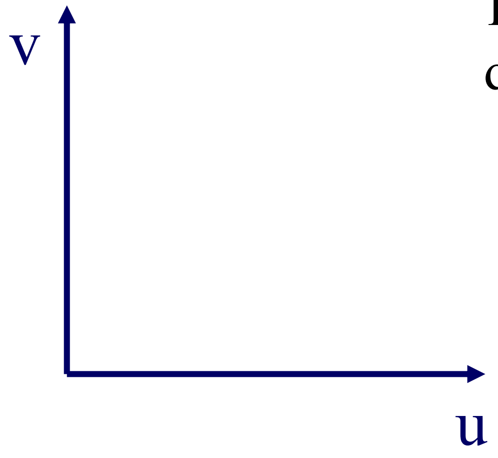


Outline

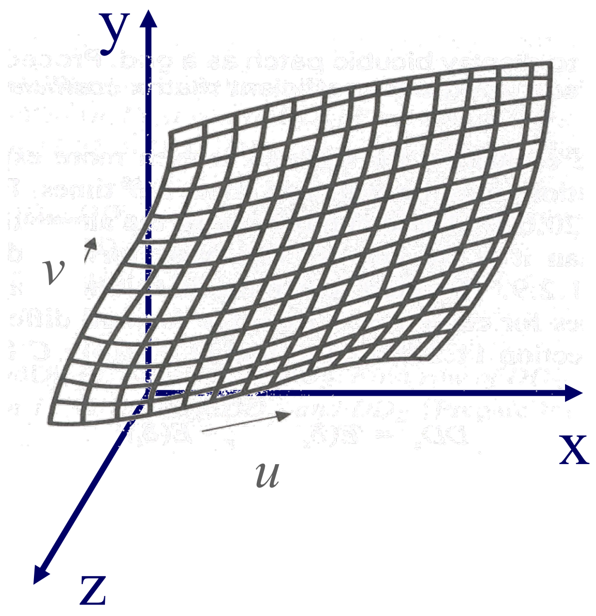
- Parametric curves
  - Cubic B-Spline
  - Cubic Bézier
- Parametric surfaces
  - Bi-cubic B-Spline
  - Bi-cubic Bézier

Parametric Surfaces

- Defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$



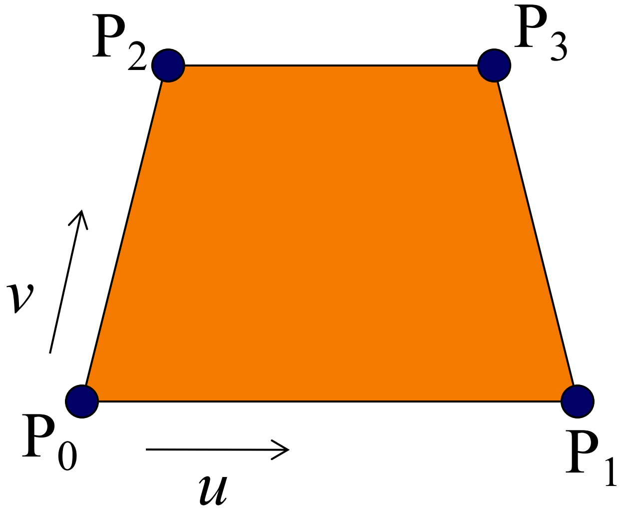
Parametric functions define mapping from (u,v) to (x,y,z):



FvDFH Figure 11.42

Parametric Surfaces

- Defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$

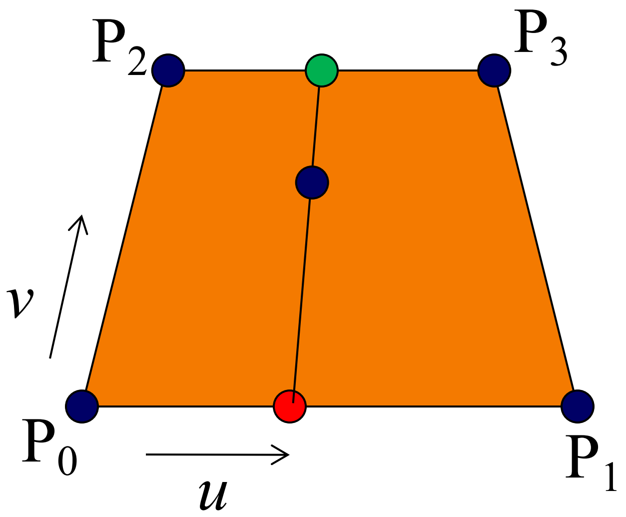


- Example: quadrilateral

$$f_x(u,v) = (1-v)\big((1-u)x_0 + ux_1\big) + v\big((1-u)x_2 + ux_3\big)$$
$$f_y(u,v) = (1-v)\big((1-u)y_0 + uy_1\big) + v\big((1-u)y_2 + uy_3\big)$$
$$f_z(u,v) = (1-v)\big((1-u)z_0 + uz_1\big) + v\big((1-u)z_2 + uz_3\big)$$

Parametric Surfaces

- Defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$



- Example: quadrilateral

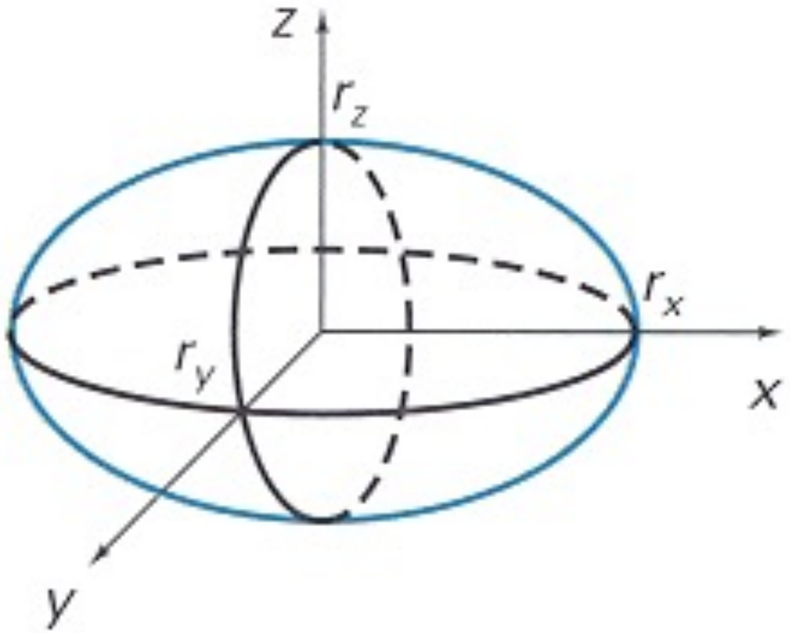
$$f_x(u,v) = (1-v)\boxed{\big((1-u)x_0 + ux_1\big)} + v\boxed{\big((1-u)x_2 + ux_3\big)}$$
$$f_y(u,v) = (1-v)\boxed{\big((1-u)y_0 + uy_1\big)} + v\boxed{\big((1-u)y_2 + uy_3\big)}$$
$$f_z(u,v) = (1-v)\boxed{\big((1-u)z_0 + uz_1\big)} + v\boxed{\big((1-u)z_2 + uz_3\big)}$$

Parametric Surfaces

- Defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$

- Example: ellipsoid

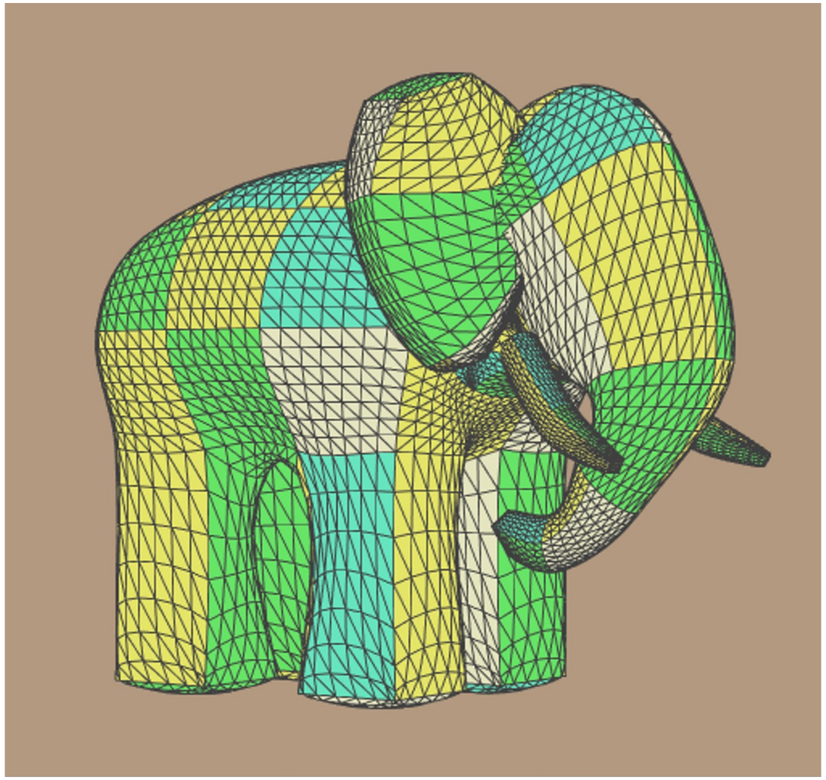
$$f_x(u,v) = r_x \cos v \cos u$$
$$f_y(u,v) = r_y \cos v \sin u$$
$$f_z(u,v) = r_z \sin v$$



H&B Figure 10.10

Parametric Surfaces

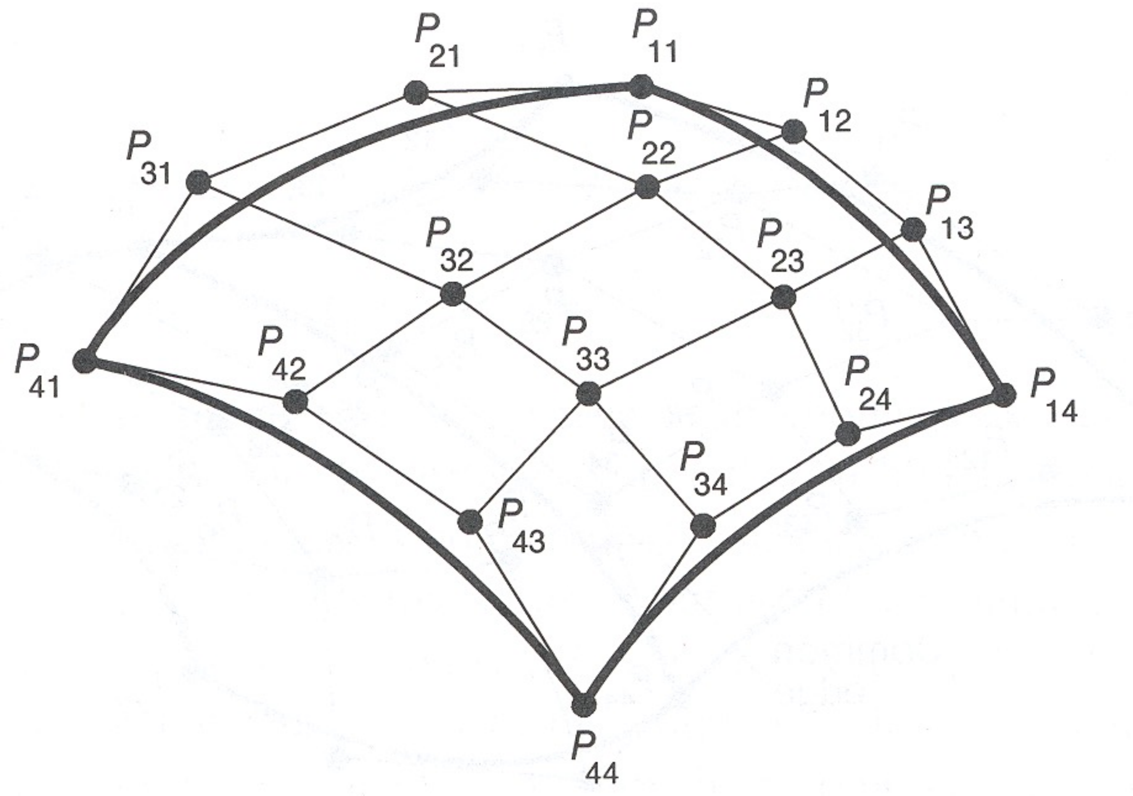
- To model arbitrary shapes, surface is partitioned into parametric *patches*



Catmull

Parametric Patches

- Each patch is defined by blending control points

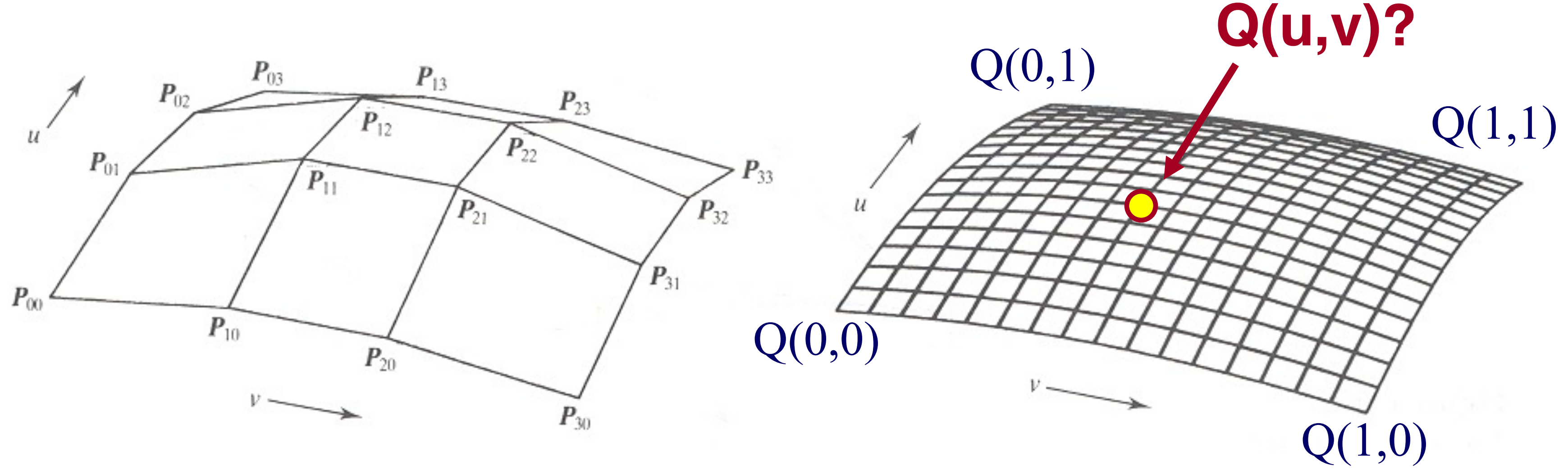


Same ideas as parametric curves!

FvDFH Figure 11.44

Parametric Patches

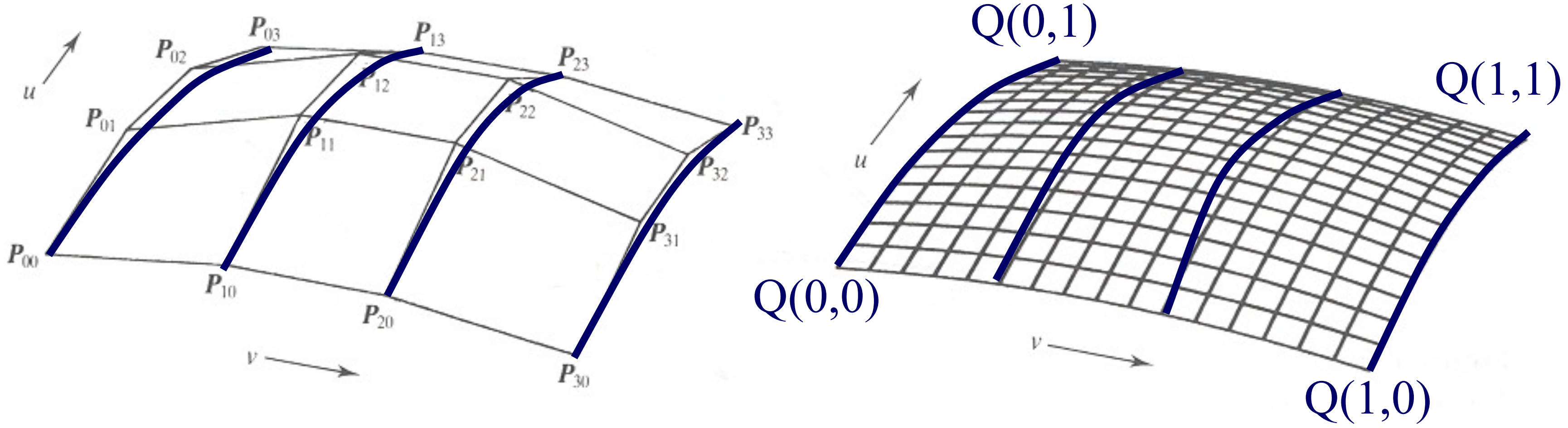
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Patches

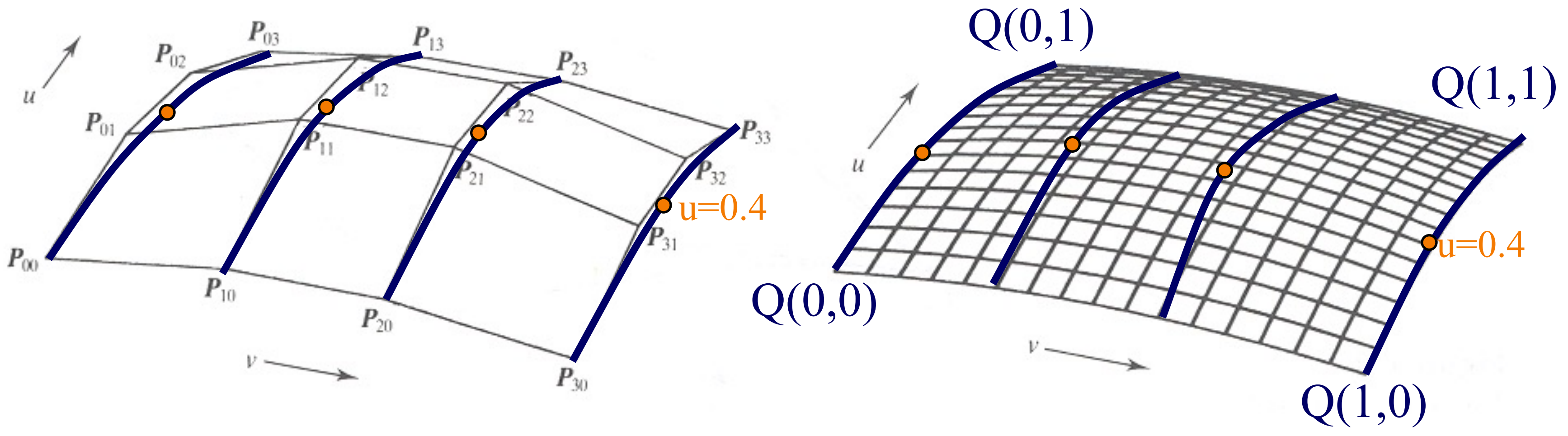
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Patches

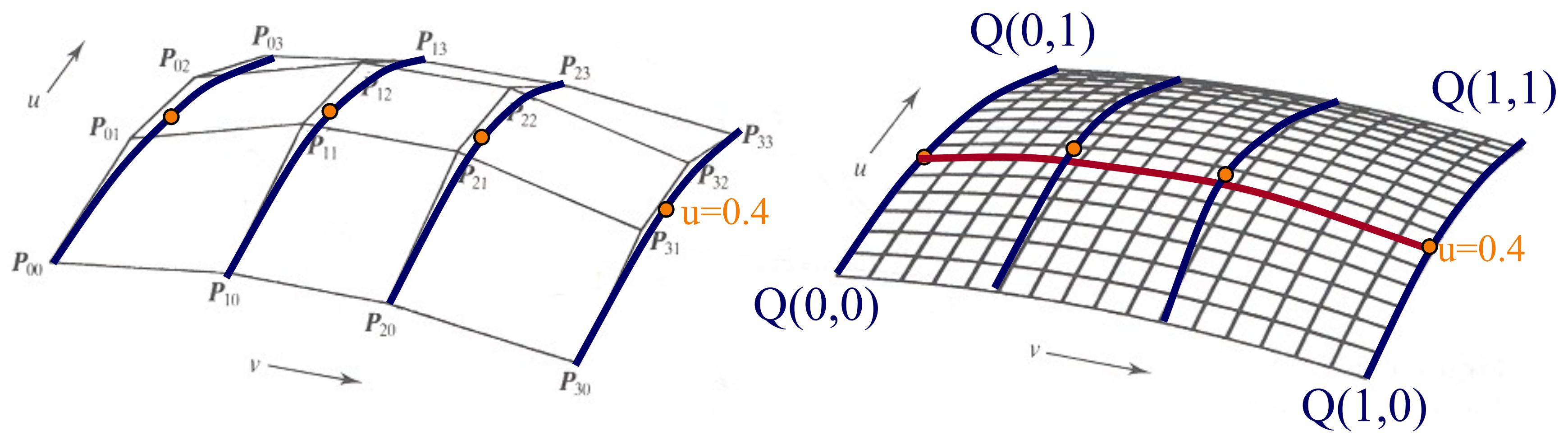
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Patches

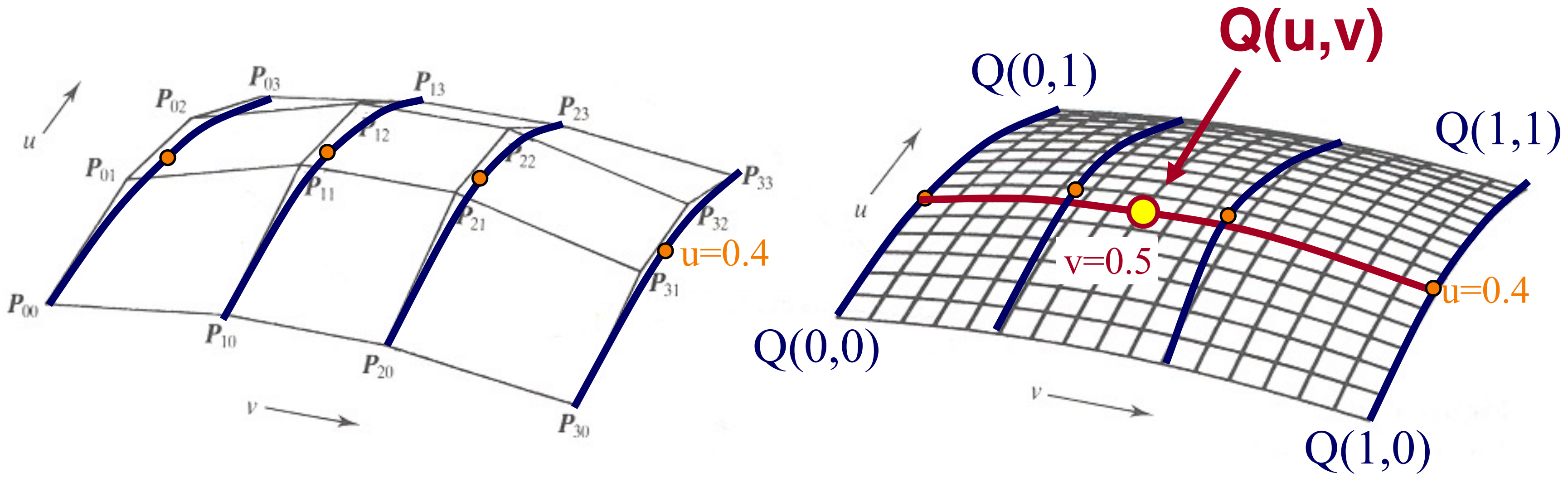
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Patches

- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Bicubic Patches

- Point Q(u,v) on any patch is defined by combining control points with polynomial blending functions:

$$Q(u,v) = \mathbf{UM}\begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix}\mathbf{M}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}$$

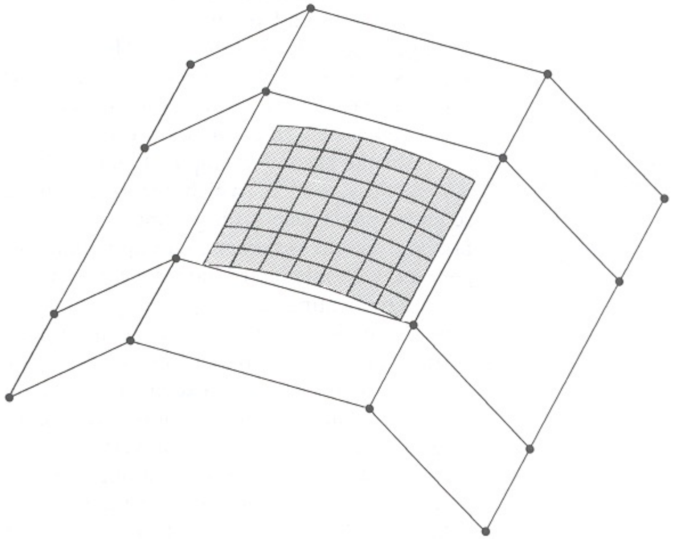
$$\mathbf{U} = [u^3 \quad u^2 \quad u \quad 1] \qquad \mathbf{V} = [v^3 \quad v^2 \quad v \quad 1]$$

Where M is a matrix describing the blending functions for a parametric cubic curve (e.g., Bézier, B-spline, etc.)

B-Spline Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\text{B-Spline}} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} \mathbf{M}^{\mathrm{T}}_{\text{B-Spline}} \mathbf{V}$$

$$\mathbf{M}_{\text{B-Spline}} = \begin{bmatrix} -1/6 & 1/2 & -1/2 & 1/6 \\ 1/2 & -1 & 1/2 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1/6 & 2/3 & 1/6 & 0 \end{bmatrix}$$

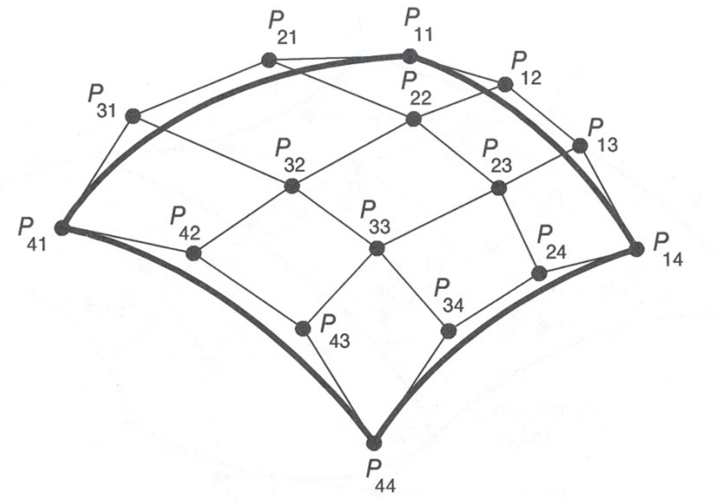


Watt Figure 6.28

Bézier Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\mathbf{Bezier}} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} \mathbf{M}_{\mathbf{Bezier}}^{\mathbf{T}} \mathbf{V}$$

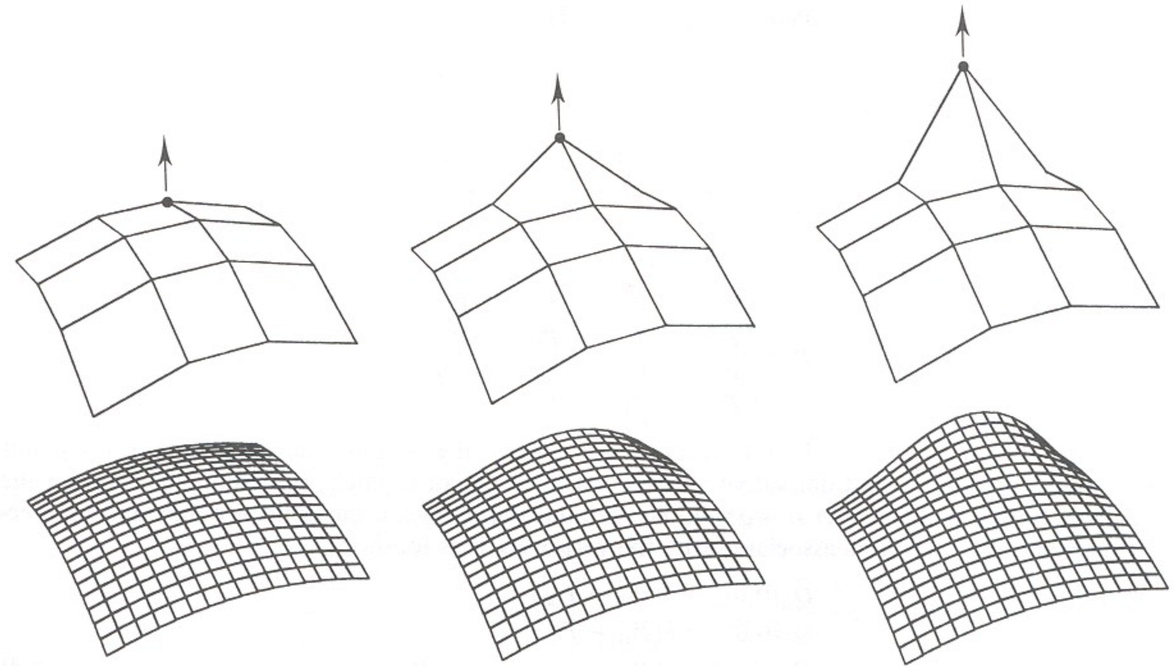
$$\mathbf{M}_{\mathbf{Bezier}} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



FvDFH Figure 11.42

Bézier Patches

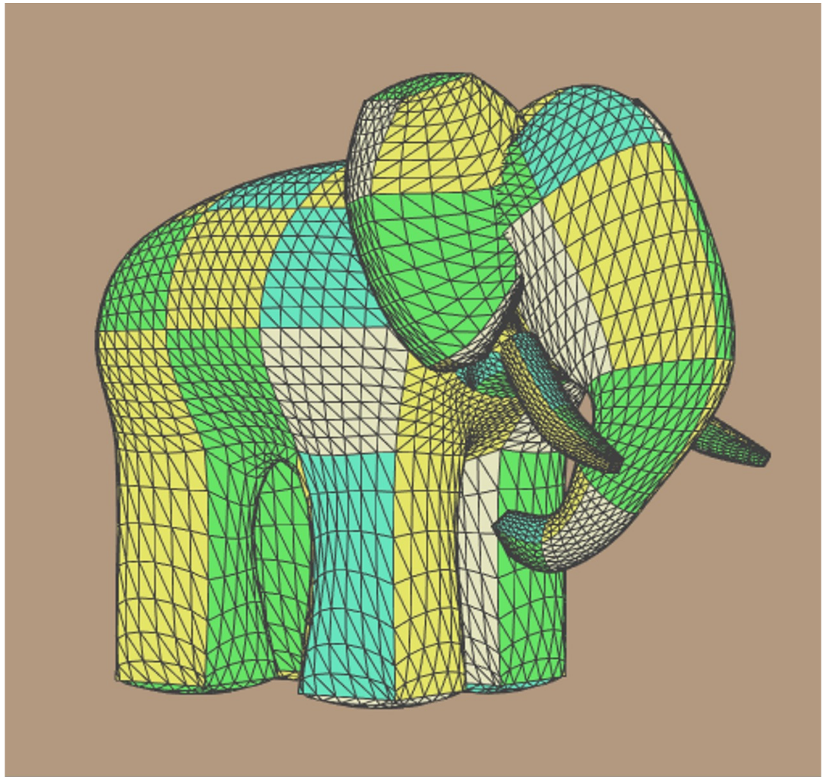
- Properties:
  - Interpolates four corner points
  - Convex hull
  - Local control



Watt Figure 6.22

Piecewise Polynomial Parametric Surfaces

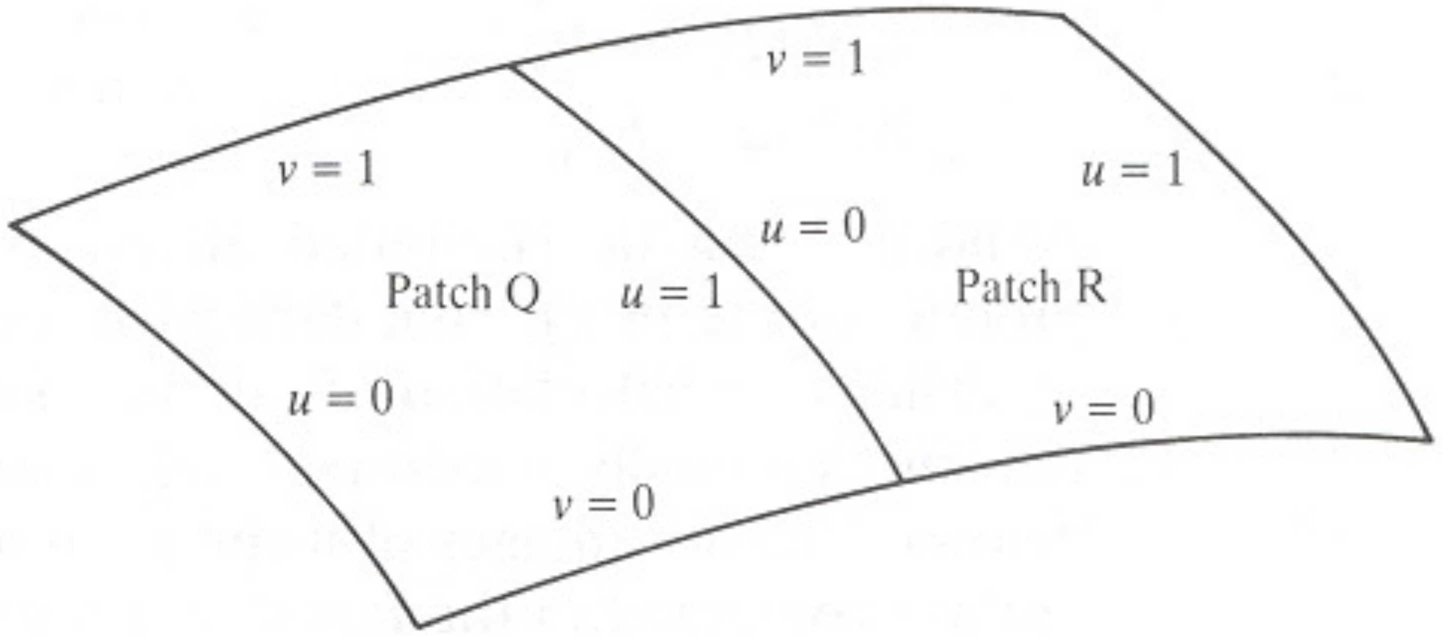
- Surface is composition of many parametric patches



Catmull

Piecewise Polynomial Parametric Surfaces

- Must maintain continuity across seams

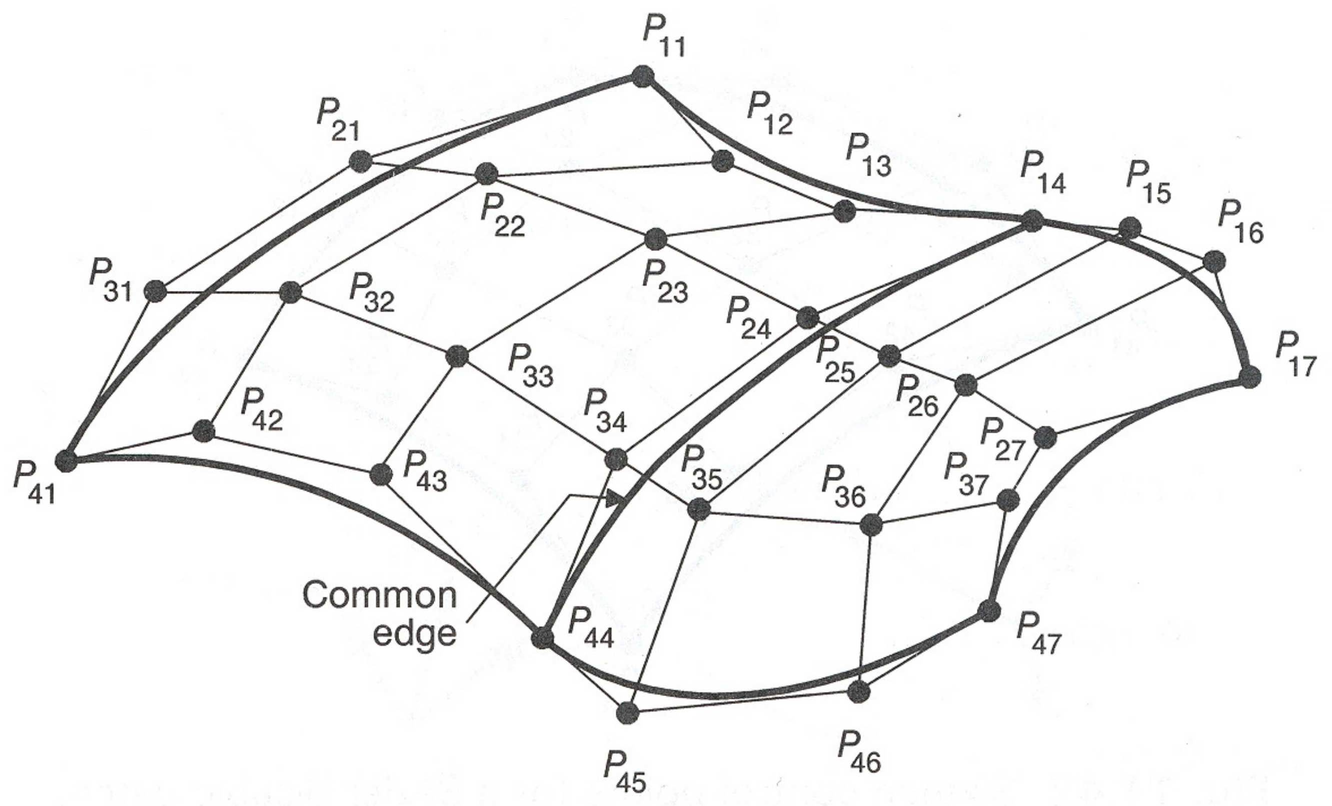


Same ideas as parametric splines!

Watt Figure 6.25

Bézier Surfaces

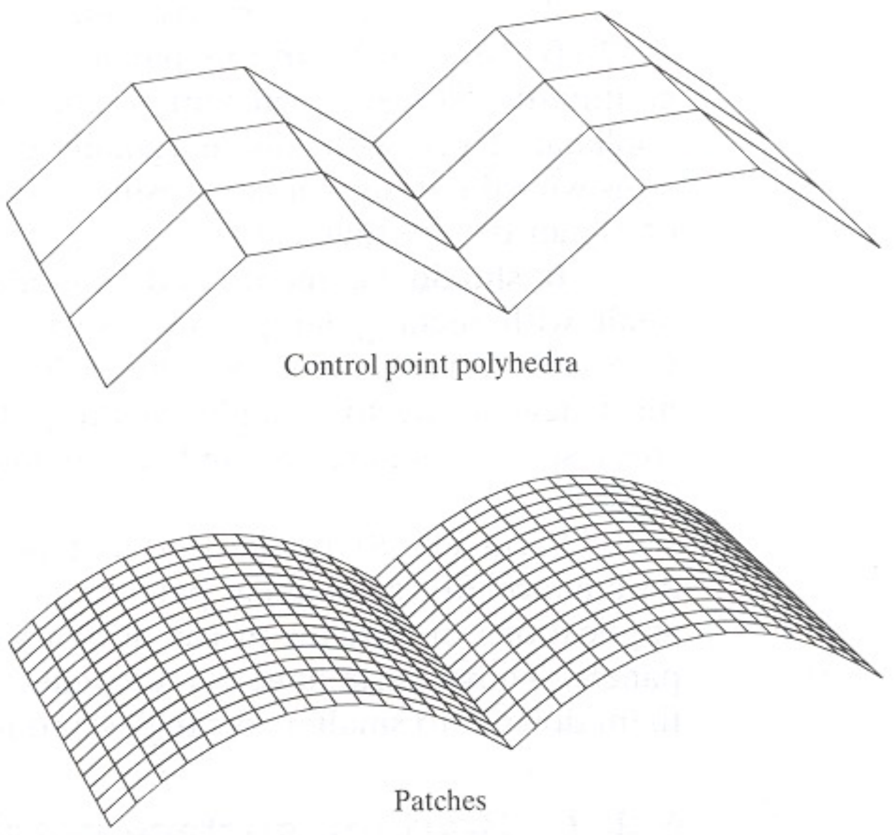
- Continuity constraints are similar to the ones for Bézier splines



FvDFH Figure 11.43

Bézier Surfaces

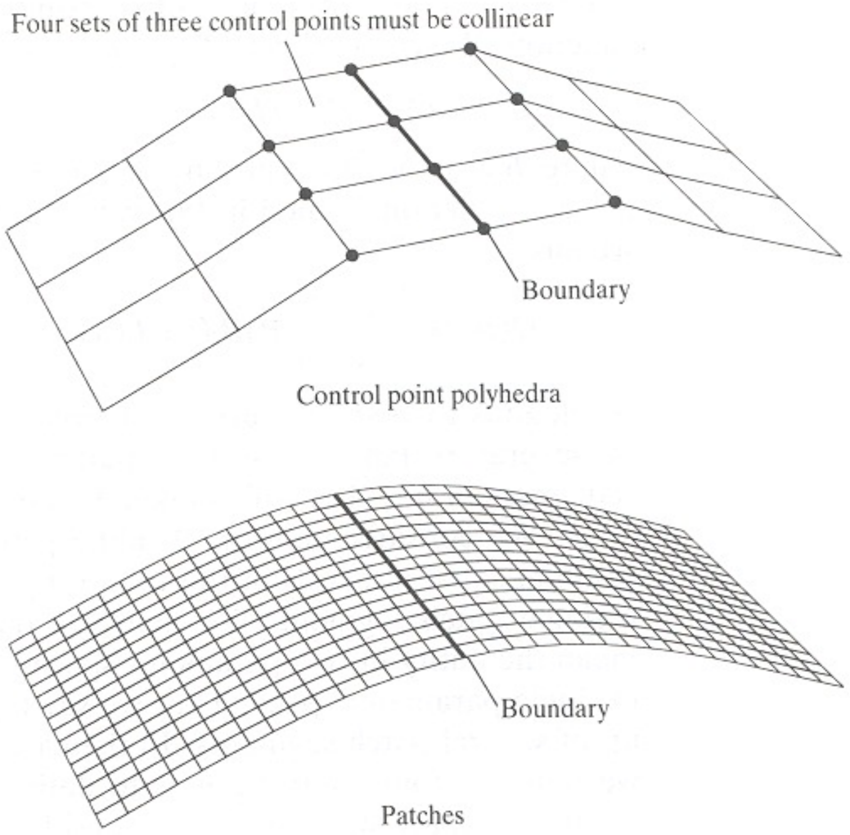
- C^0 continuity requires aligning boundary curves



Watt Figure 6.26a

Bézier Surfaces

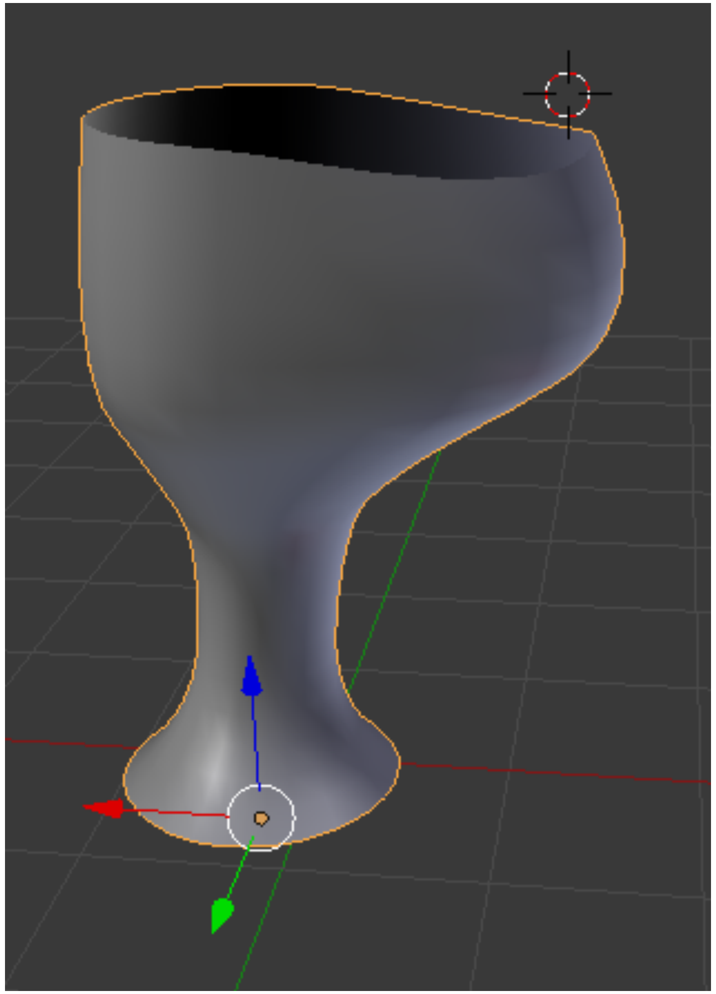
- C^1 continuity requires aligning boundary curves and derivatives



Watt Figure 6.26b

NURBS

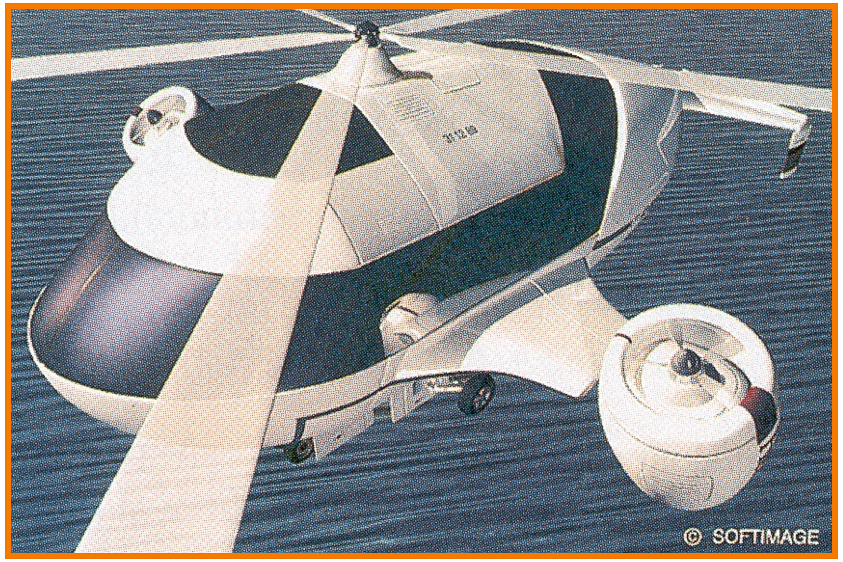
- **N**on-**U**niform **R**ational **B**-**S**plines
- Most common parametric surface representation
- Per-control-point weights, non-uniform parameterization provide greater control and artistic flexibility



blender

Parametric Surfaces

- Properties
  - ? Natural parameterization
  - ? Guaranteed smoothness
  - ? Intuitive editing
  - ? Concise
  - ? Accurate
  - ? Efficient display
  - ? Easy acquisition
  - ? Efficient intersections
  - ? Guaranteed validity
  - ? Arbitrary topology



Parametric Surfaces

- Properties
  - ☺ Natural parameterization
  - ☺ Guaranteed smoothness
  - ☺ Intuitive editing
  - ☺ Concise
  - ☺ Accurate
  - o Efficient display
  - ☹ Easy acquisition
  - ☹ Efficient intersections
  - ☹ Guaranteed validity
  - ☹ Arbitrary topology