More on Transformations

COS 426, Fall 2022

PRINCETON UNIVERSITY

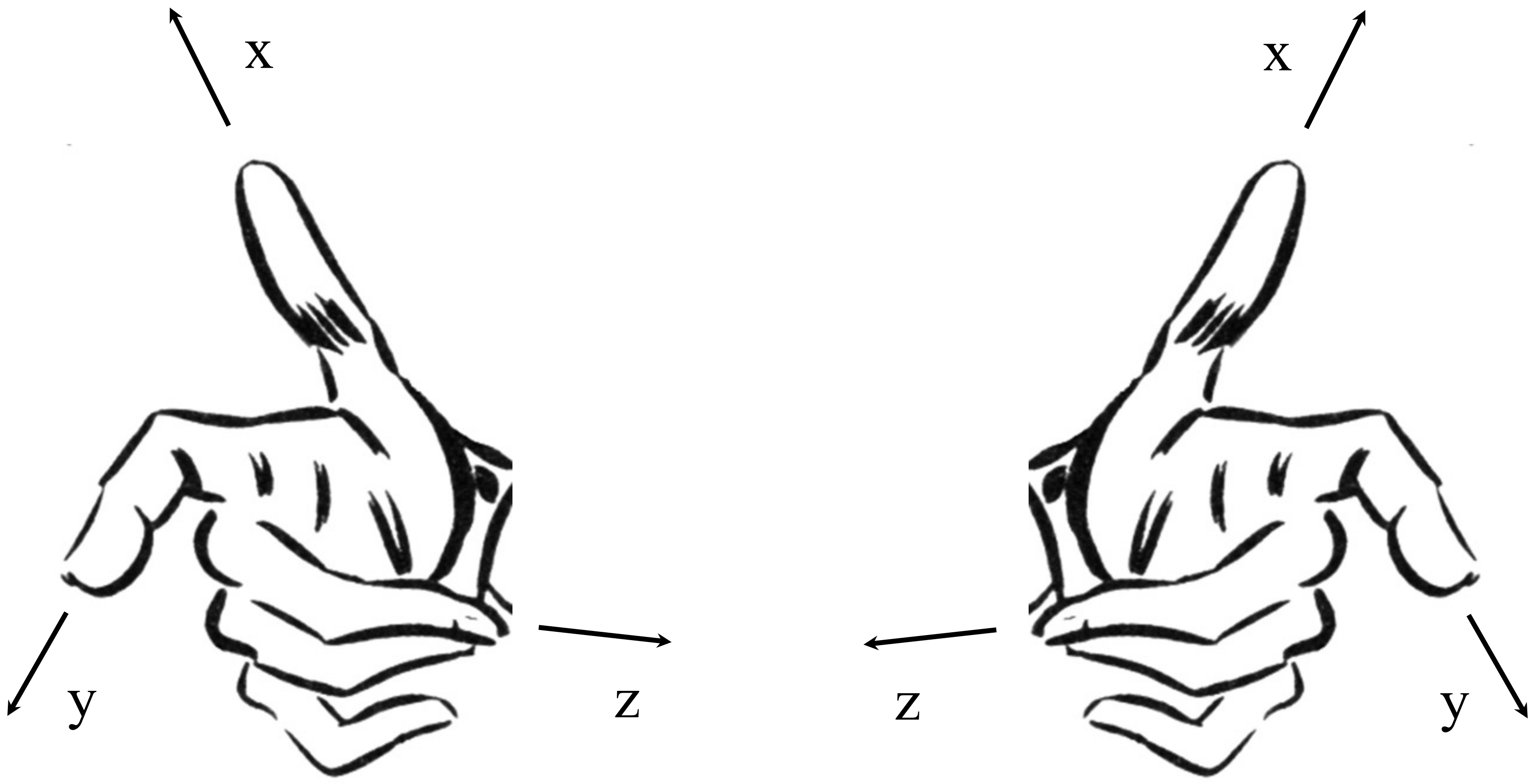
Agenda

Grab-bag of topics related to transformations:

- General rotations
  - Euler angles
  - Rodrigues's rotation formula
- Maintaining camera transformations
  - First-person
  - Trackball
- How to transform normals

3D Coordinate Systems

- Right-handed vs. left-handed



3D Coordinate Systems

- Right-handed vs. left-handed
- Right-hand rule for rotations:
  positive rotation = counterclockwise rotation about axis

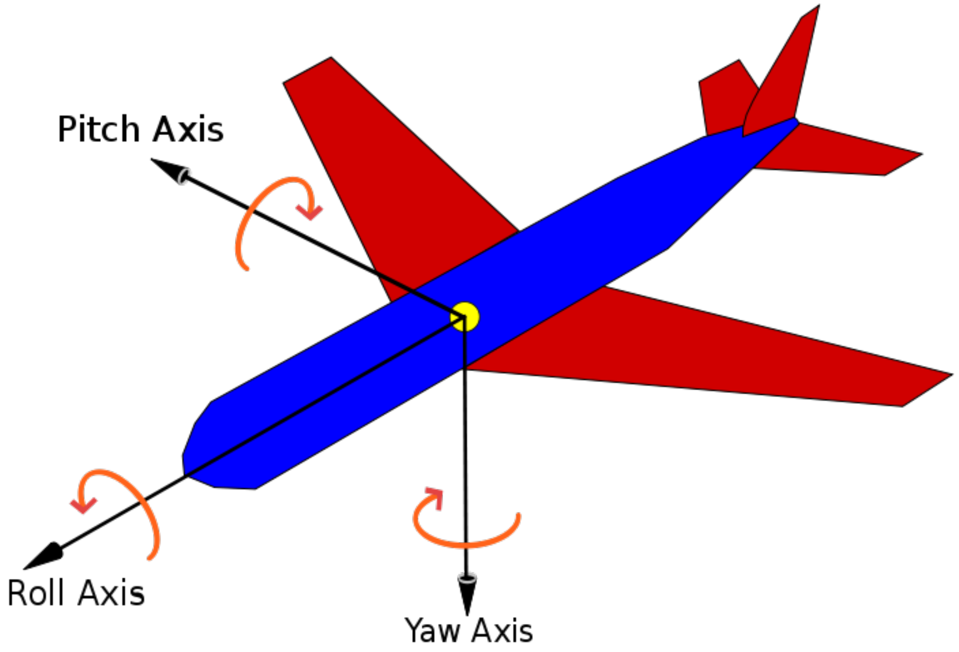
General Rotations

- Recall: set of rotations in 3-D is 3-dimensional
  - Rotation group SO(3)
  - Non-commutative
  - Corresponds to orthonormal 3×3 matrices with determinant = +1

- Need 3 parameters to represent a general rotation (Euler's rotation theorem)

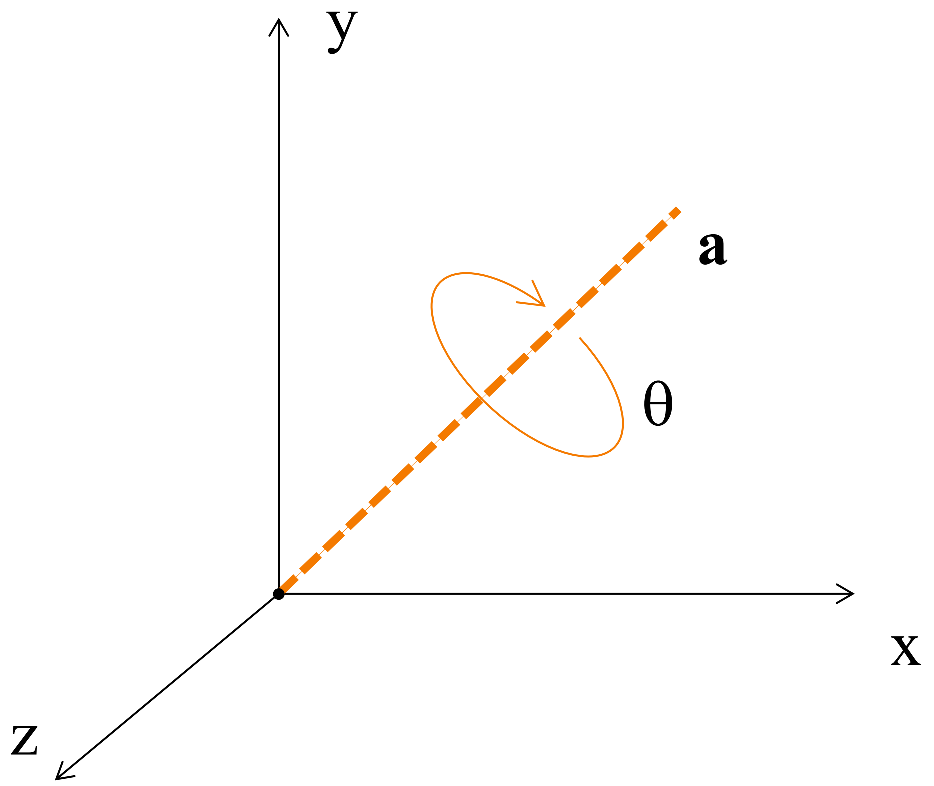
Euler Angles

- Specify rotation by giving angles of rotation about 3 coordinate axes
- 12 possible conventions for order of axes, but one standard is X-Y-Z
  - Can be interpreted as yaw, pitch, roll of airplane



Rodrigues's Formula

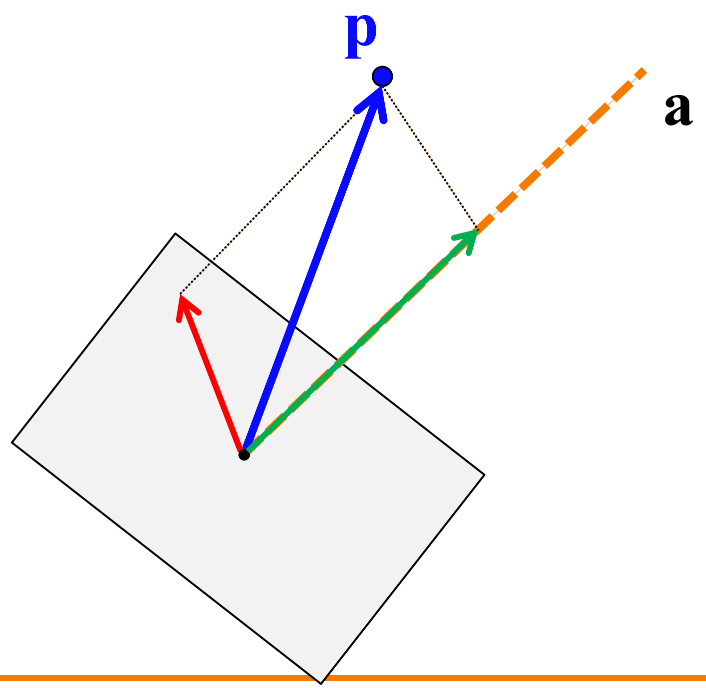
- Even more useful: rotate by an angle (1 number) about an arbitrary axis (3 numbers, but only 2 degrees of freedom since unit-length)



Rodrigues's Formula

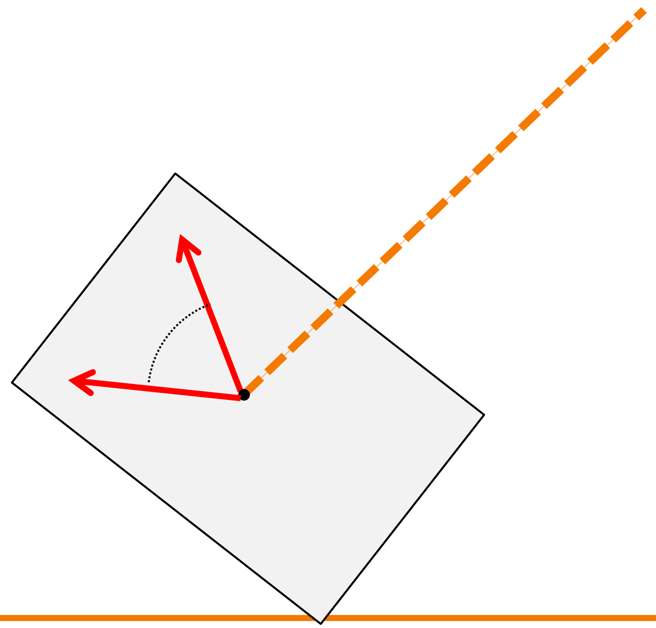
- An arbitrary point $\mathbf{p}$ may be decomposed into its components along and perpendicular to $\mathbf{a}$

$$\mathbf{p} = \mathbf{a}\,(\mathbf{p} \cdot \mathbf{a}) + [\mathbf{p} - \mathbf{a}\,(\mathbf{p} \cdot \mathbf{a})]$$



Rodrigues's Formula

- Rotating component along **a** leaves it unchanged
- Rotating component perpendicular to **a** (call it $\mathbf{p}_\perp$) moves it to $\mathbf{p}_\perp \cos\theta + (\mathbf{a} \times \mathbf{p}_\perp) \sin\theta$



Rodrigues's Formula

- Putting it all together:

$$\mathbf{R}\mathbf{p} = \mathbf{a}\,(\mathbf{p} \cdot \mathbf{a}) + \mathbf{p}_\perp \cos\theta + (\mathbf{a} \times \mathbf{p}_\perp) \sin\theta$$

Why?

$$= \mathbf{a}\mathbf{a}^T\mathbf{p} + (\mathbf{p} - \mathbf{a}\mathbf{a}^T\mathbf{p}) \cos\theta + (\mathbf{a} \times \mathbf{p}) \sin\theta$$

- So,

$$\mathbf{R} = \mathbf{a}\mathbf{a}^T + (\mathbf{I} - \mathbf{a}\mathbf{a}^T) \cos\theta + [\mathbf{a}]_\times \sin\theta$$

where $[\mathbf{a}]_\times$ is the "cross product matrix"

$$[\mathbf{a}]_\times = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}$$

Rotating One Direction into Another

- Given two directions $\mathbf{d}_1$, $\mathbf{d}_2$ (unit length), how to find transformation that rotates $\mathbf{d}_1$ into $\mathbf{d}_2$?
  - There are many such rotations!
  - Choose rotation with minimum angle
- Axis = $\mathbf{d}_1 \times \mathbf{d}_2$
- Angle = $\mathrm{acos}(\mathbf{d}_1 \cdot \mathbf{d}_2)$
- More stable numerically: $\mathrm{atan2}(|\mathbf{d}_1 \times \mathbf{d}_2|, \mathbf{d}_1 \cdot \mathbf{d}_2)$

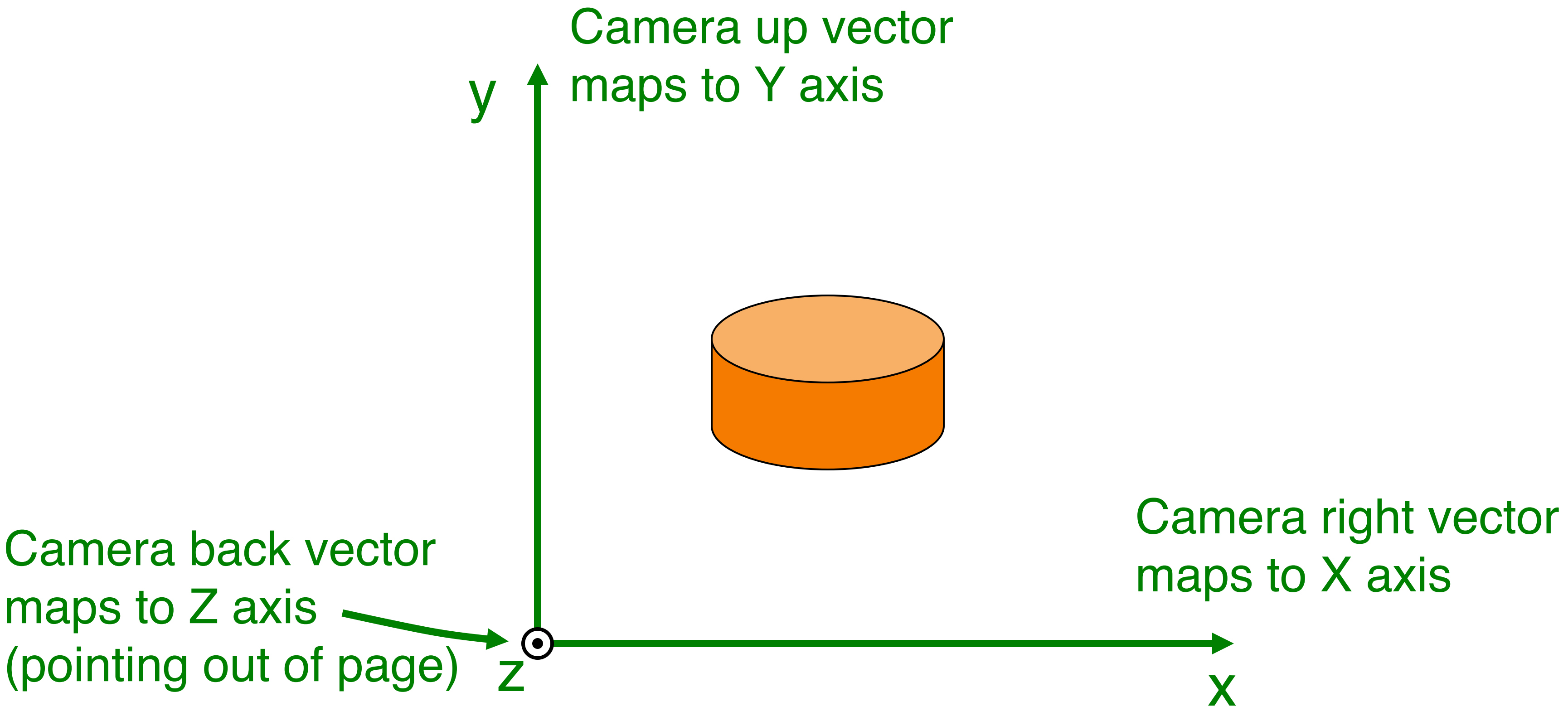
Agenda

Grab-bag of topics related to transformations:

- General rotations
  - Euler angles
  - Rodrigues's rotation formula
- Maintaining camera transformations
  - First-person
  - Trackball
- How to transform normals

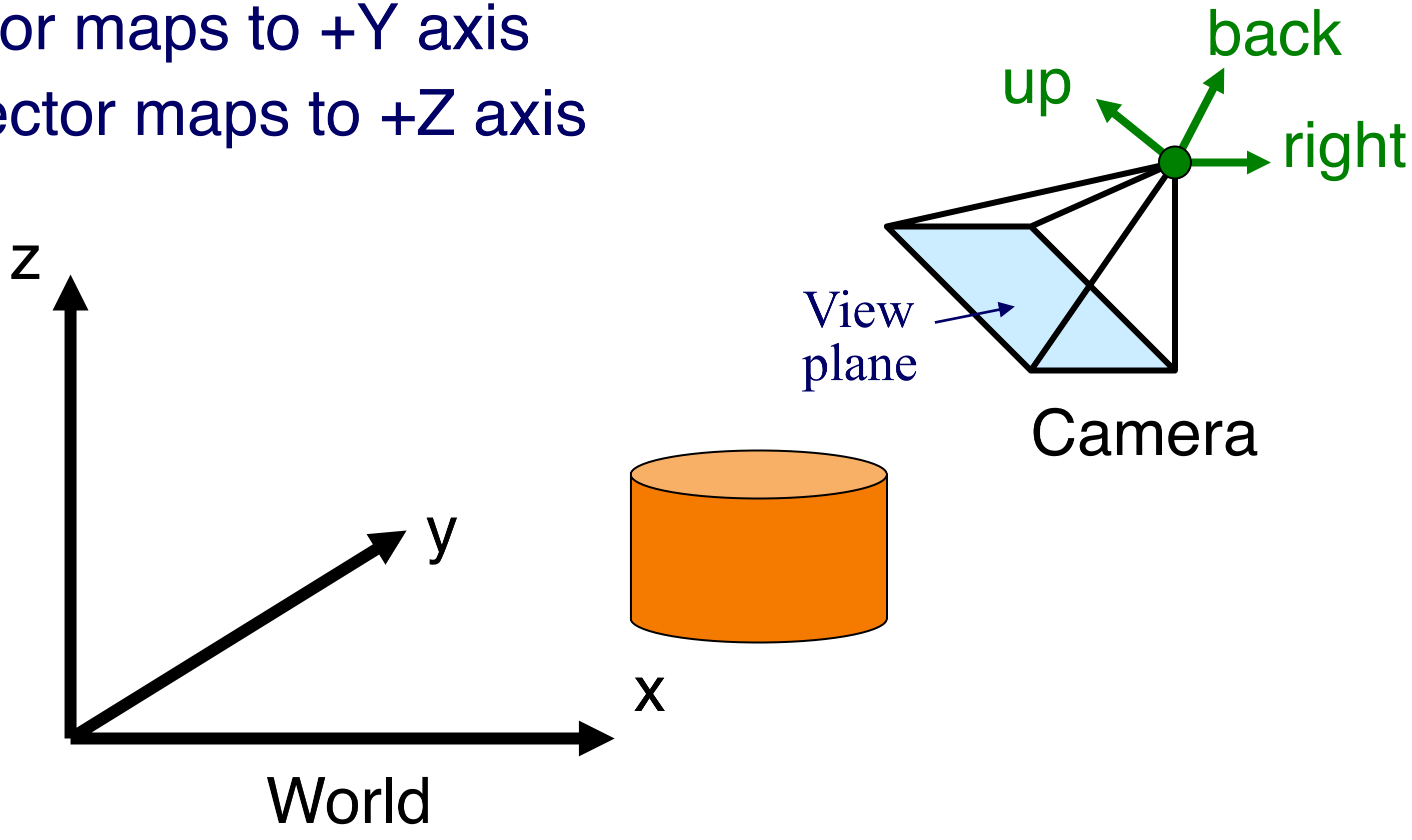
Camera Coordinates

- Canonical camera coordinate system
  - Convention is right-handed (looking down –z axis)
  - Convenient for projection, clipping, etc.



Viewing Transformation

- Mapping from world to camera coordinates
  - Eye position maps to origin
  - Right vector maps to +X axis
  - Up vector maps to +Y axis
  - Back vector maps to +Z axis



Finding the viewing transformation

- We have the camera (in world coordinates)
- We want T taking objects from world to camera

$$p^c = T\ p^w$$

- Trick: find T^{-1} taking objects in camera to world

$$p^w = T^{-1} p^c$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

?

Finding the Viewing Transformation

- Trick: map from camera coordinates to world
  - Origin maps to eye position
  - Z axis maps to Back vector
  - Y axis maps to Up vector
  - X axis maps to Right vector

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} R_x & U_x & B_x & E_x \\ R_y & U_y & B_y & E_y \\ R_z & U_z & B_z & E_z \\ R_w & U_w & B_w & E_w \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

- This matrix is T^{-1} so we invert it to get T … easy!

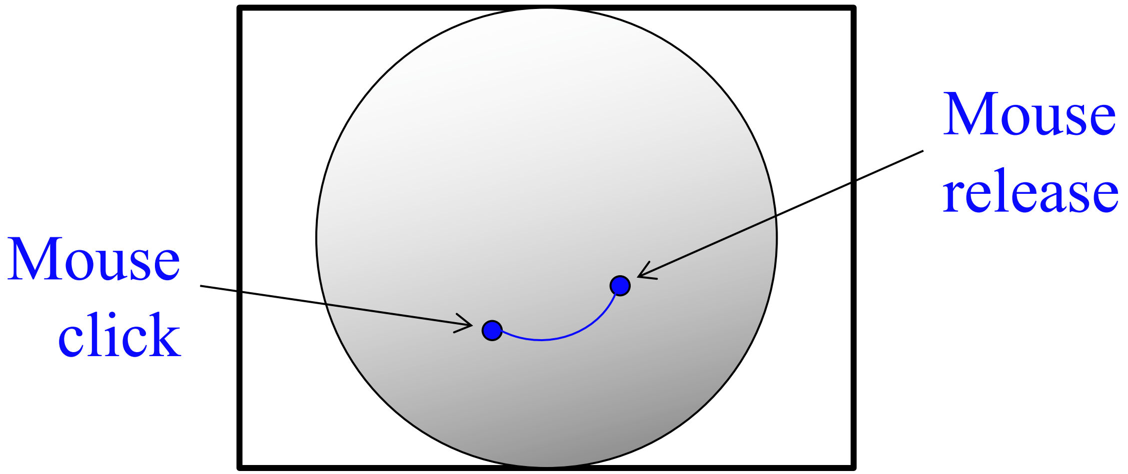
Maintaining Viewing Transformation

- For first-person camera control, need 2 operations:
  - Turn: rotate(θ, 0,1,0) in local coordinates
  - Advance: translate(0, 0, –v*Δt) in local coordinates
- Key: transformations act on local, not global coords
  - To accomplish: right-multiply by translation, rotation

$$\mathbf{M}_{new} \leftarrow \mathbf{M}_{old}\mathbf{T}_{-v^*\Delta t,z}\mathbf{R}_{\theta,y}$$

Maintaining Viewing Transformation

- Object manipulation: “trackball” or “arcball” interface
  - Map mouse positions to surface of a sphere



  - Compute rotation axis, angle
  - Apply rotation to global coords: left-multiply

$$\mathbf{M}_{new} \leftarrow \mathbf{R}_{\theta,a}\,\mathbf{M}_{old}$$

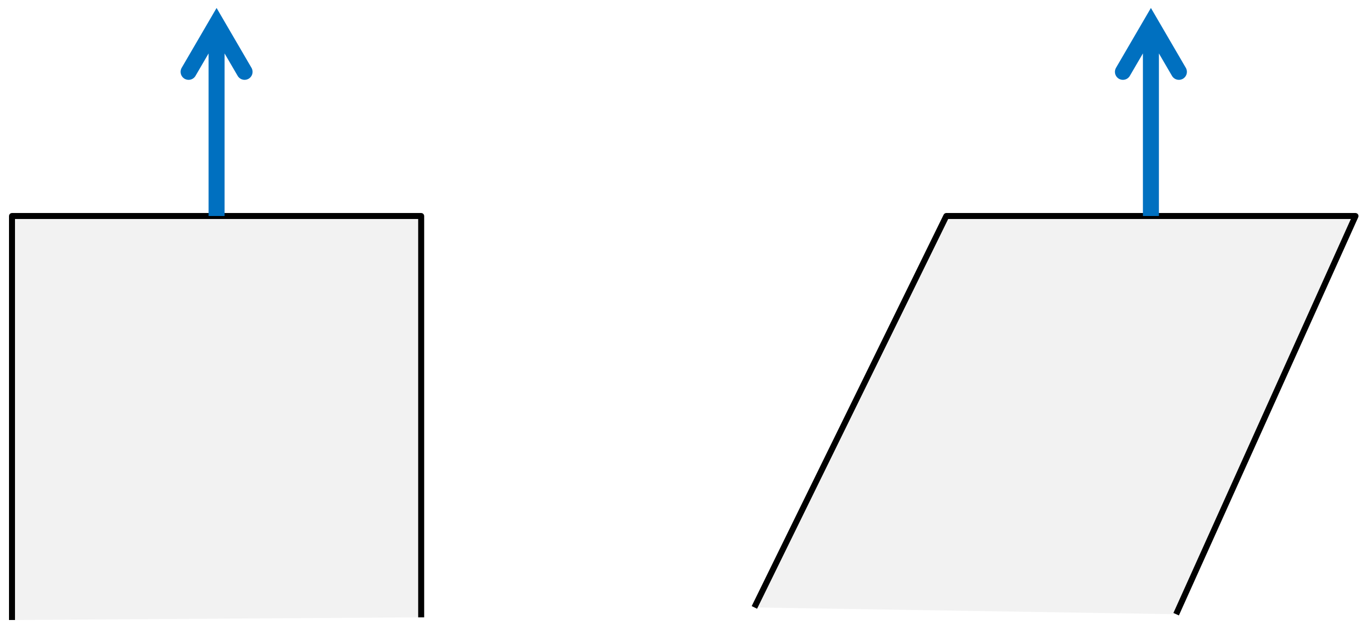
Agenda

Grab-bag of topics related to transformations:

- General rotations
  - Euler angles
  - Rodrigues's rotation formula
- Maintaining camera transformations
  - First-person
  - Trackball
- How to transform normals

Transforming Normals

- Normals do not transform the same way as points!
  - Not affected by translation
  - Not affected by shear perpendicular to the normal



Transforming Normals

- Key insight: normal remains perpendicular to surface tangent

- Let **t** be a tangent vector and **n** be the normal

$$\mathbf{t} \cdot \mathbf{n} = 0 \qquad \text{or} \qquad \mathbf{t}^T\mathbf{n} = 0$$

- If matrix **M** represents an affine transformation, it transforms **t** as

$$\mathbf{t} \to \mathbf{M}_L\mathbf{t}$$

where $\mathbf{M}_L$ is the linear part (upper-left 3×3) of **M**

Transforming Normals

- So, after transformation, want

$$(\mathbf{M_L t})^T \mathbf{n}_{\text{transformed}} = 0$$

- But we know that

$$\mathbf{t}^T \mathbf{n} = 0$$

$$\mathbf{t}^T \mathbf{M_L}^T (\mathbf{M_L}^T)^{-1} \mathbf{n} = 0$$

$$(\mathbf{M_L t})^T (\mathbf{M_L}^T)^{-1} \mathbf{n} = 0$$

- So,

$$\mathbf{n}_{\text{transformed}} = (\mathbf{M_L}^T)^{-1} \mathbf{n}$$

Transforming Normals

- Conclusion: normals transformed by *inverse transpose* of *linear part* of transformation

- Note that for rotations, inverse = transpose, so inverse transpose = identity
  - normals just rotated

COS 426 Midterm exam

- This Thursday, Oct 13
- Completed online in Gradescope (access through Canvas)
- Available 3 PM – midnight (but we'll be most active monitoring for questions, which you should post as *private* messages on Ed, during the first part of that range)
- Covers everything through last week: color, image processing, shape representations, transformations (but not today's lecture)
- You may refer to materials on course website, but "closed Internet"

COS 426 Midterm Q/A

Additional Midterm Q/A in lecture.