An OCaml definition of OCaml evaluation, or,

Implementing OCaml in OCaml

COS 326
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Defining Programming Language Semantics

To write a program, you have to know how the language works.

Semantics: The study of "how a programming language works"

Methods for defining program semantics:

- Operational: show how to rewrite program expressions step-by-step until you end up with a value
  - we've done some of this already
- Denotational: how interpret a program in a different language that is well understood
  - we aren't going to do much of this – see COS 510
- Equational: specify the equal programs
  - we'll do more of this later & use this semantics to prove things about our programs
- Axiomatic: provide (other kinds of) reasoning rules about programs

Defining Program Semantics

Today, we'll focus on operational definitions

We'll use the following techniques to communicate:

1. *examples* (good for intuition, but highly incomplete)
   – this doesn't get at the corner cases
2. *an interpreter program* written in OCaml
3. *mathematical notation*

Defining Program Semantics

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   – this doesn't get at the corner cases
2. *an interpreter program* written in OCaml
3. *mathematical notation*

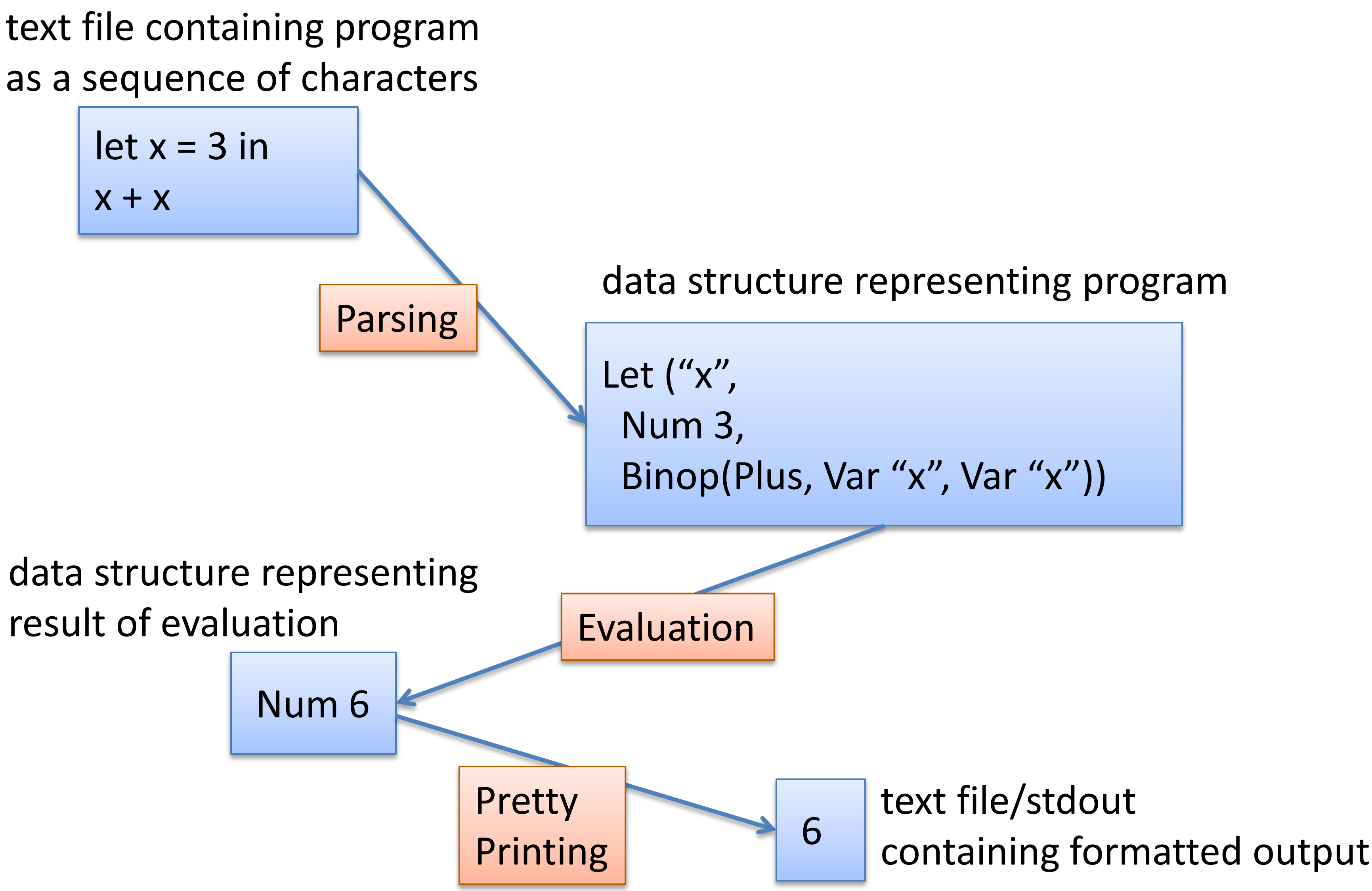
our focus today

PRELIMINARIES

Reading: Note on “Operational Semantics”

https://www.cs.princeton.edu/courses/archive/fall19/cos326/notes/evaluation.php

Implementing an Interpreter



REPRESENTING SYNTAX

Representing Syntax

Program syntax is a complicated tree-like data structure.

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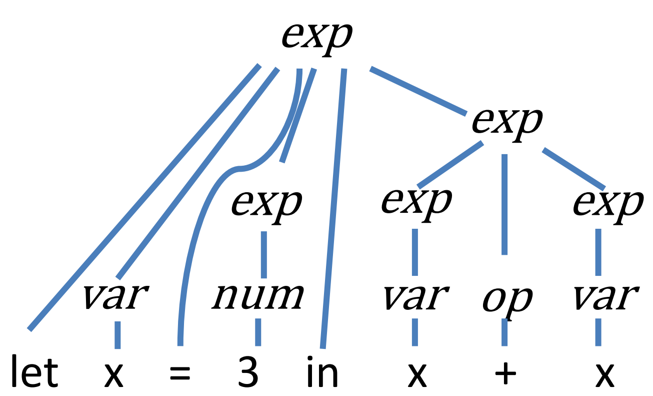
```
let x = 3 in
x + x
```

Syntax Trees

let x = 3 in x + x

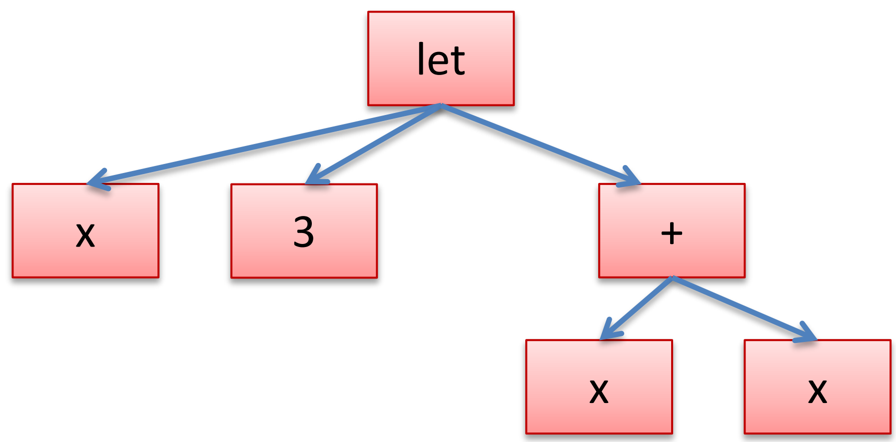


This is the "parse tree." Useful for some purposes, but for the semantics it's Too Much Information.

Abstract Syntax Tree (AST)

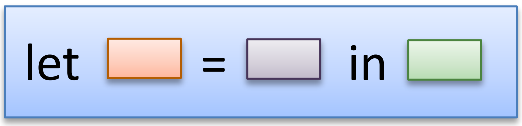
Don’t need to represent all the “punctuation”

```
let x = 3 in
x + x
```



Representing Syntax

More generally each let expression has 3 parts:



Representing Syntax

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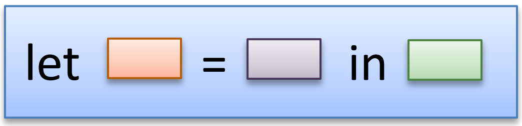
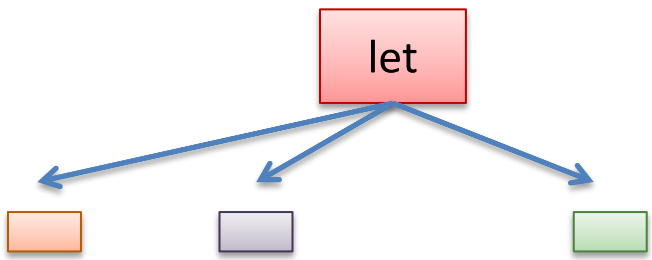


And you can represent a let expression using a tree like this:



Representing Syntax

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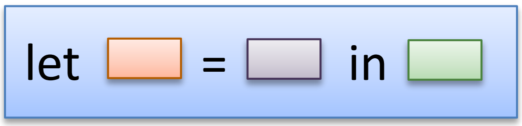

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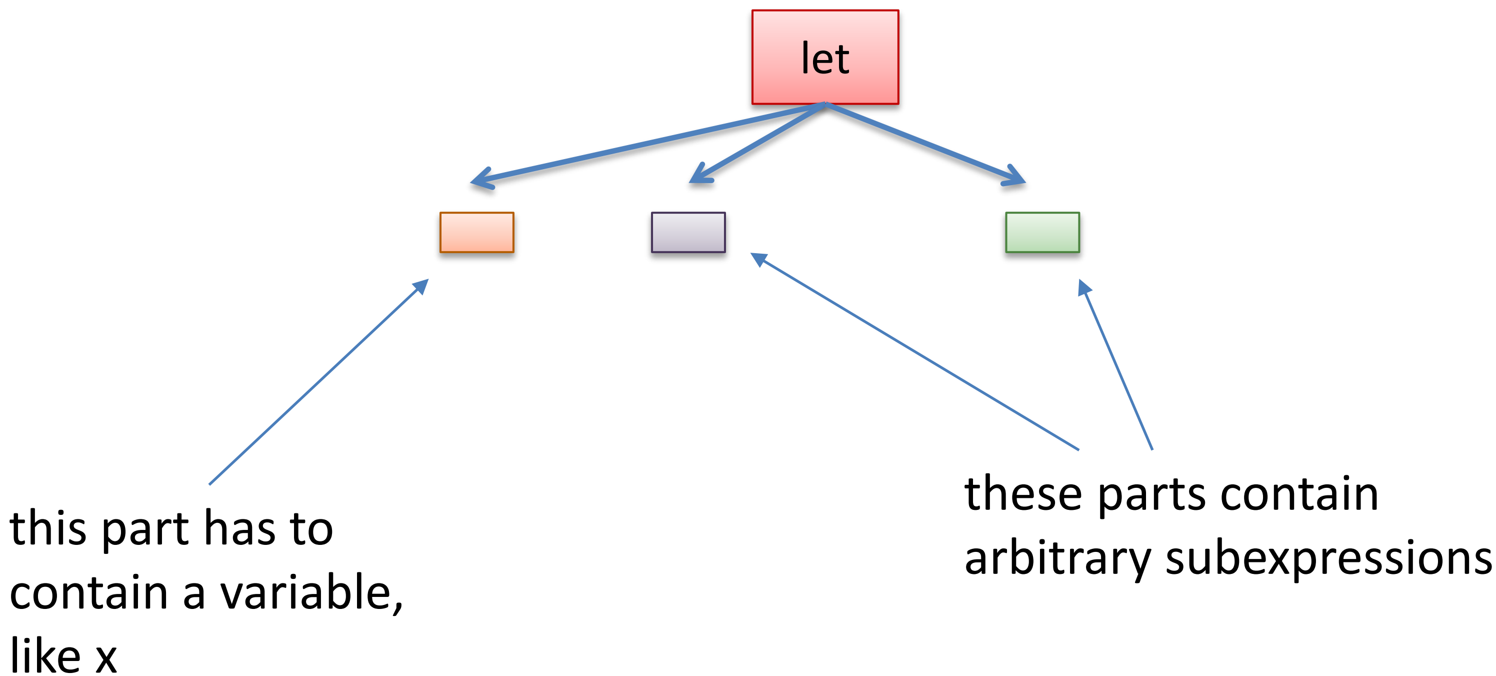
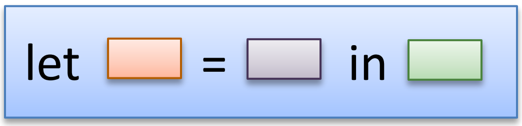
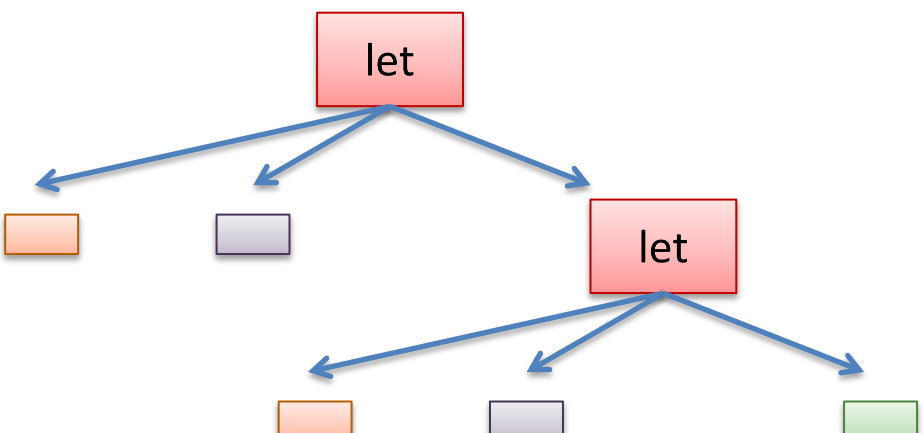


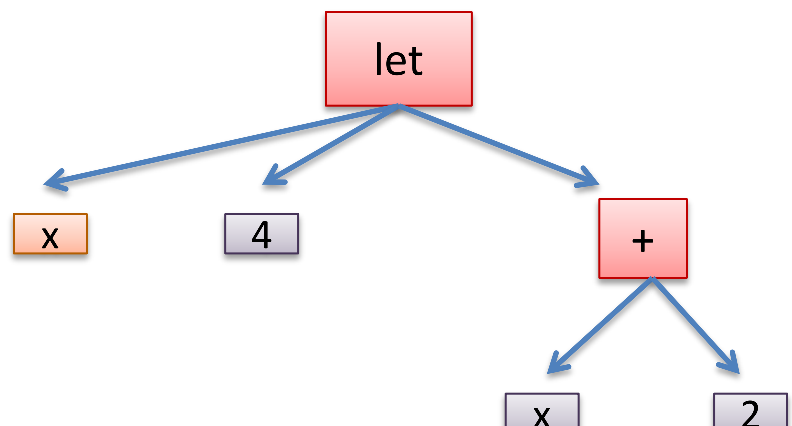
Representing Syntax

More generally each let expression has 3 parts:



And you create complicated programs by nesting let expressions (or any other expression) recursively inside one another:





OCaml for the Win

Functional programming languages have sometimes been called "domain-specific languages for compiler writers"

Datatypes are amazing for representing complicated tree-like structures and that is exactly what a program is.

Use a different constructor for every different sort of expression

- one constructor for variables
- one constructor for let expressions
- one constructor for numbers
- one constructor for binary operators, like add
- ...

Aside: Java for the loss

Languages like Java, that are based exclusively around heavy-weight class tend to be vastly more verbose when trying to represent syntax trees:

- one whole class for each different kind of syntax
- one class for variables
- one class for let expressions
- one class for numbers ...

In addition, writing traversals over the syntax is annoying, because your code is spread over N different classes (using a visitor pattern) rather than in one place.

Aside: Java for the loss

Languages like Java, that are based exclusively around heavy-weight class tend to be vastly more verbose when trying to represent syntax trees:

- one whole class for each d... ...ntax
- one class for
- o...
- one cl...

SCORE: OCAML 3.8, JAVA 0

(C: who cares?)

In addi... ...annoy...
becaus... ...ere... ...sses (using a visitor pattern) ... one p...ce.

Making These Ideas Precise

A datatype for simple OCaml expressions:

```
type variable = string

type op = Plus | Minus | Times | …

type exp =
  | Int_e of int
  | Op_e  of exp * op * exp
  | Var_e of variable
  | Let_e of variable * exp * exp

type value = exp
```

Making These Ideas Precise

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  | Let_e of variable * exp * exp
type value = exp

let e1 = Int_e 3
```

Making These Ideas Precise

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  | Let_e of variable * exp * exp
type value = exp

let e1 = Int_e 3
let e2 = Int_e 17
```

Making These Ideas Precise

A datatype for simple OCaml expressions:

```
type variable = string
type op = Plus | Minus | Times | …
type exp =
   | Int_e of int
   | Op_e  of exp * op * exp
   | Var_e of variable
   | Let_e of variable * exp * exp
type value = exp

let e1 = Int_e 3
let e2 = Int_e 17
let e3 = Op_e (e1, Plus, e2)
```

represents “3 + 17”

Making These Ideas Precise

We can represent the OCaml program:

```
let x = 30 in
  let y =
    (let z = 3 in
     z*4)
  in
  y+y
```

This is called
concrete syntax
(concrete syntax pertains to parsing)

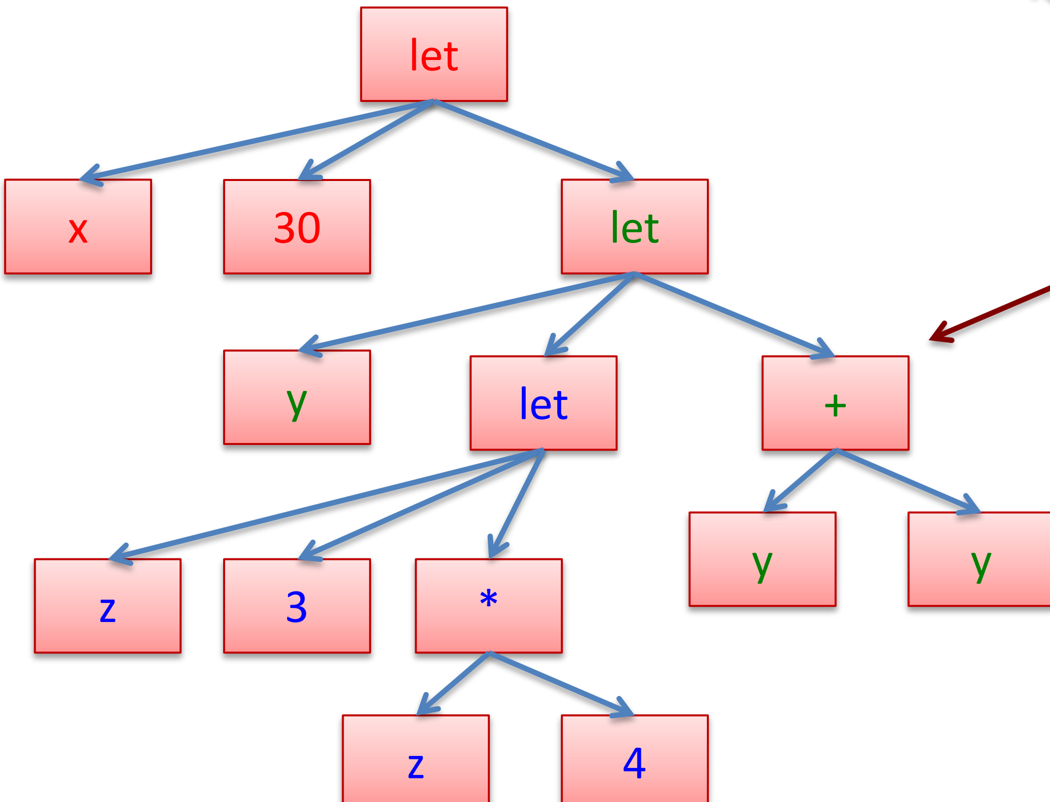
This is called an
abstract syntax tree (AST)

as an exp value:

```
Let_e("x", Int_e 30,
      Let_e("y",
            Let_e("z", Int_e 3,
                  Op_e(Var_e "z", Times, Int_e 4)),
      Op_e(Var_e "y", Plus, Var_e "y")
```

ASTs as … Trees

```
Let_e("x",Int_e 30,
      Let_e("y",Let_e("z",Int_e 3,
                    Op_e(Var_e "z", Times, Int_e 4)),
           Op_e(Var_e "y", Plus, Var_e "y")
```



Notice how the OCaml expression can be drawn as a tree

ASTs as … Trees


```
Let_e("x",Int_e 30,
      Let_e("y",Let_e("z",Int_e 3,
                      Op_e(Var_e "z", Times, Int_e 4)),
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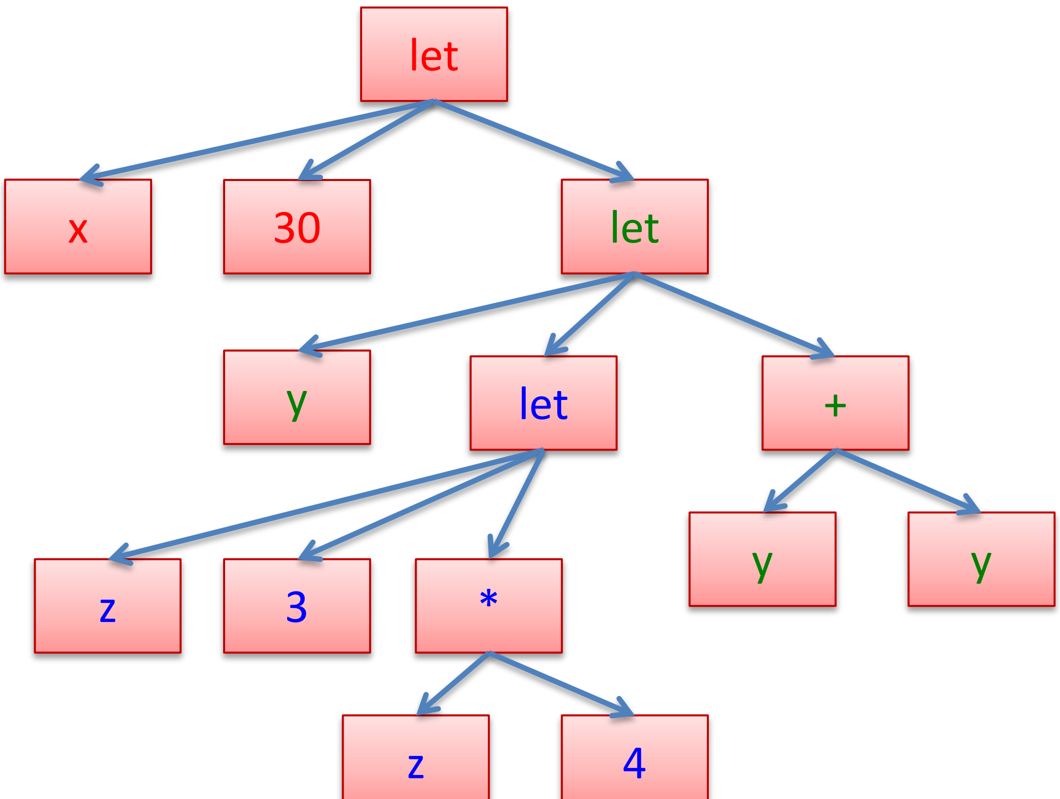
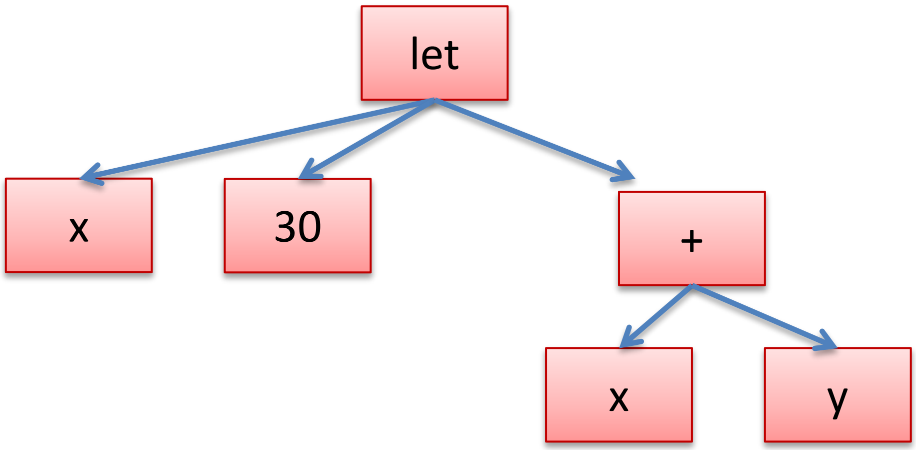
```



By thinking about programs as their abstract syntax trees we can make certain notions, like the scope of a variable, which we've talked about before, more precise.

Free vs Bound Variables


```
let x = 30 in
x+y
```



Free vs Bound Variables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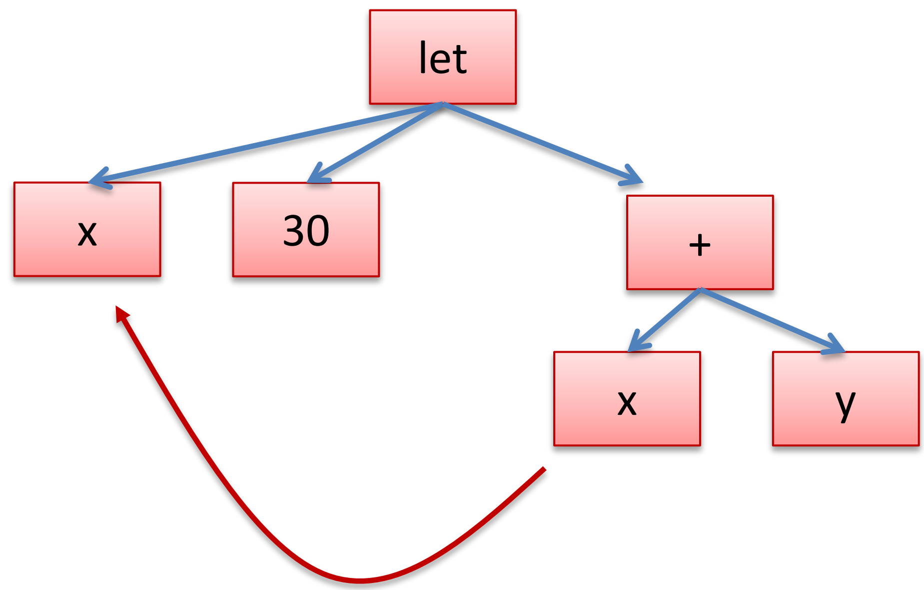

this use of x is bound here

Free vs Bound Variables

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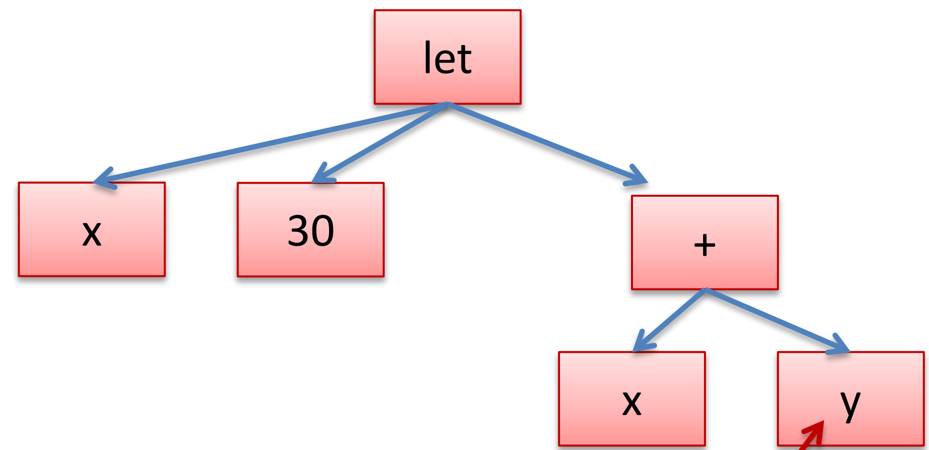


this use of y is free

we say: "y is a free variable in the expression (let x = 30 in x+y)"

Other Examples


```
fun z -> z + y
```

z is bound

y is a free variable

```
match x with
 (y,z) -> y + z + w
```

x, w are free variables

y, z are bound

```
let rec f x =
  match x with
    [] -> y
  | hd:tl -> hd::f tl
```

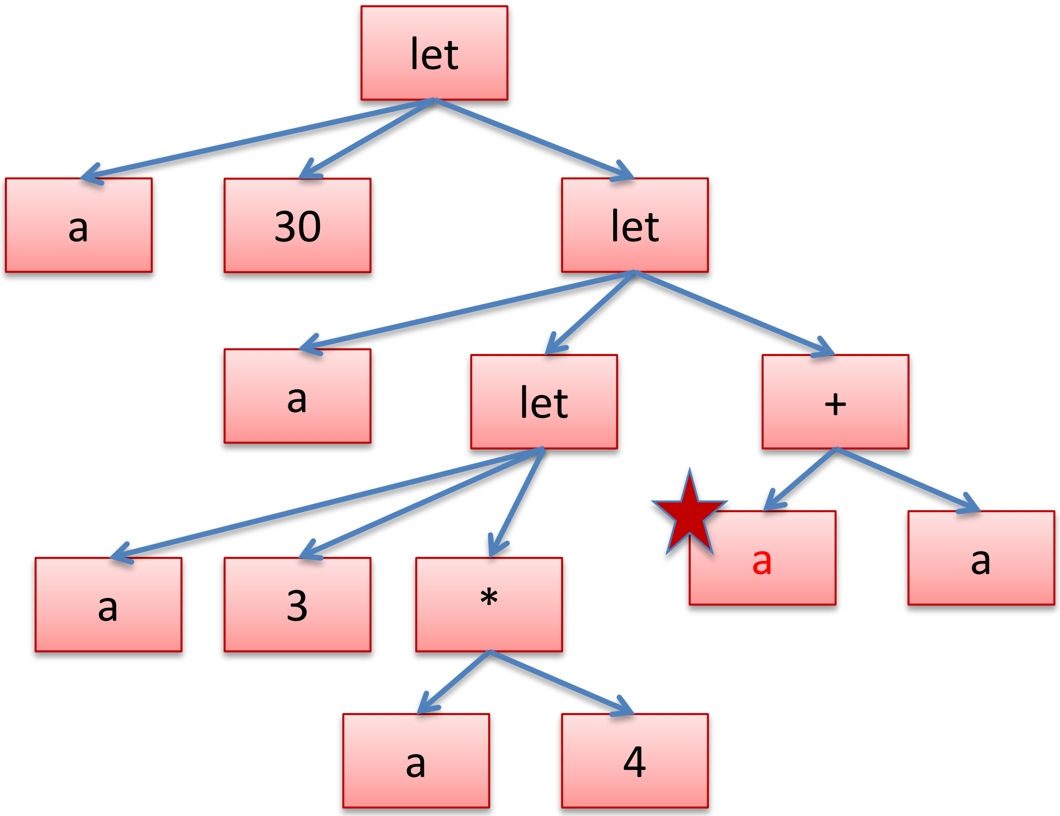
y is a free variable

f, x, hd, tl are all bound

Abstract Syntax Trees

Given a variable occurrence, we can find where it is bound by …

```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a
```



Abstract Syntax Trees

crawling up the tree to the nearest enclosing let…

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let a = 30 in
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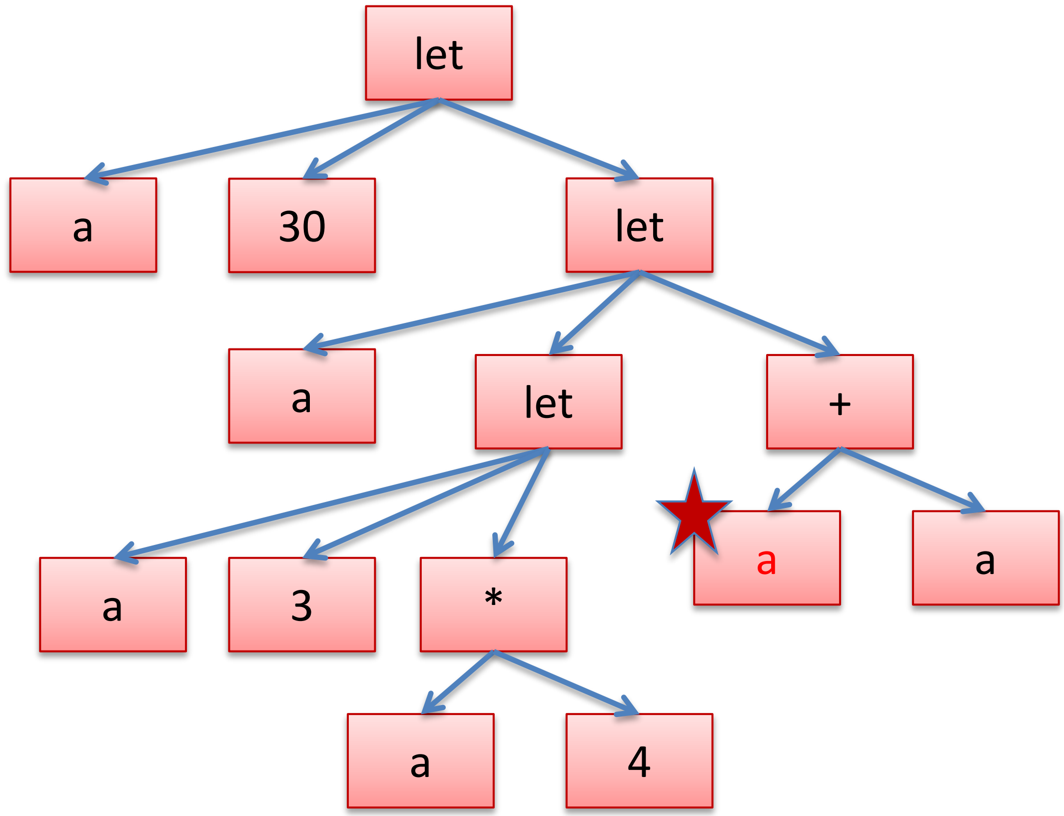
```



Abstract Syntax Trees

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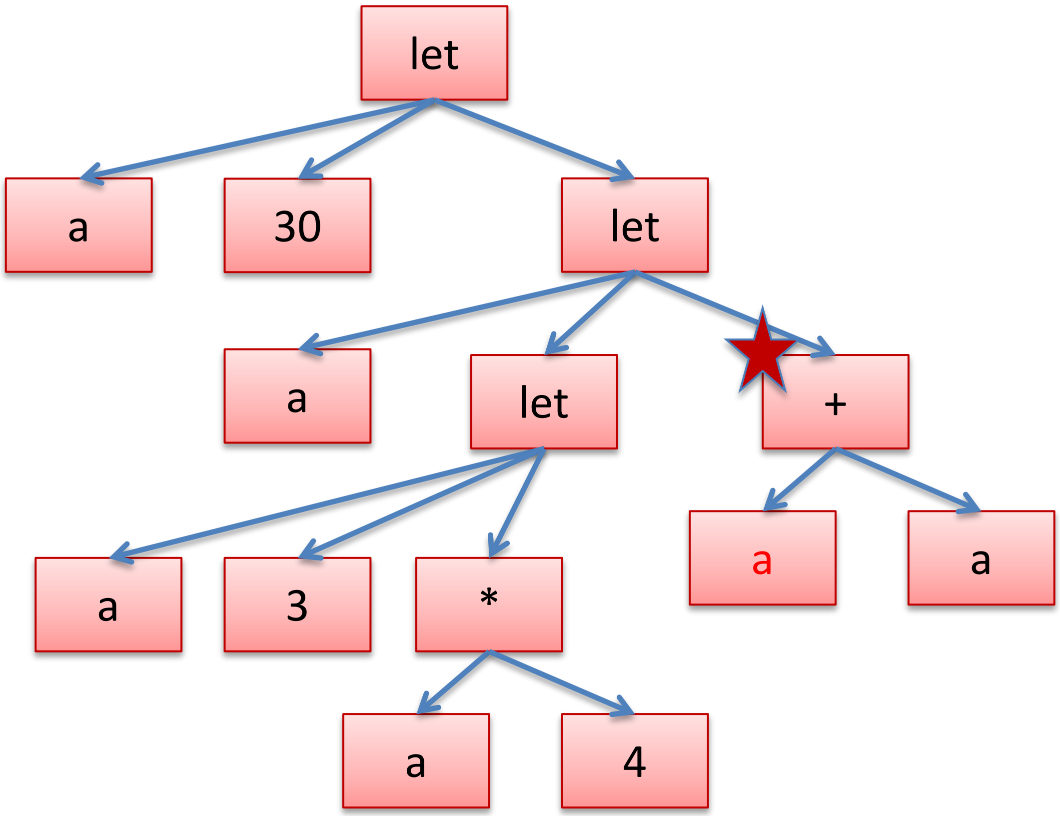

Abstract Syntax Trees

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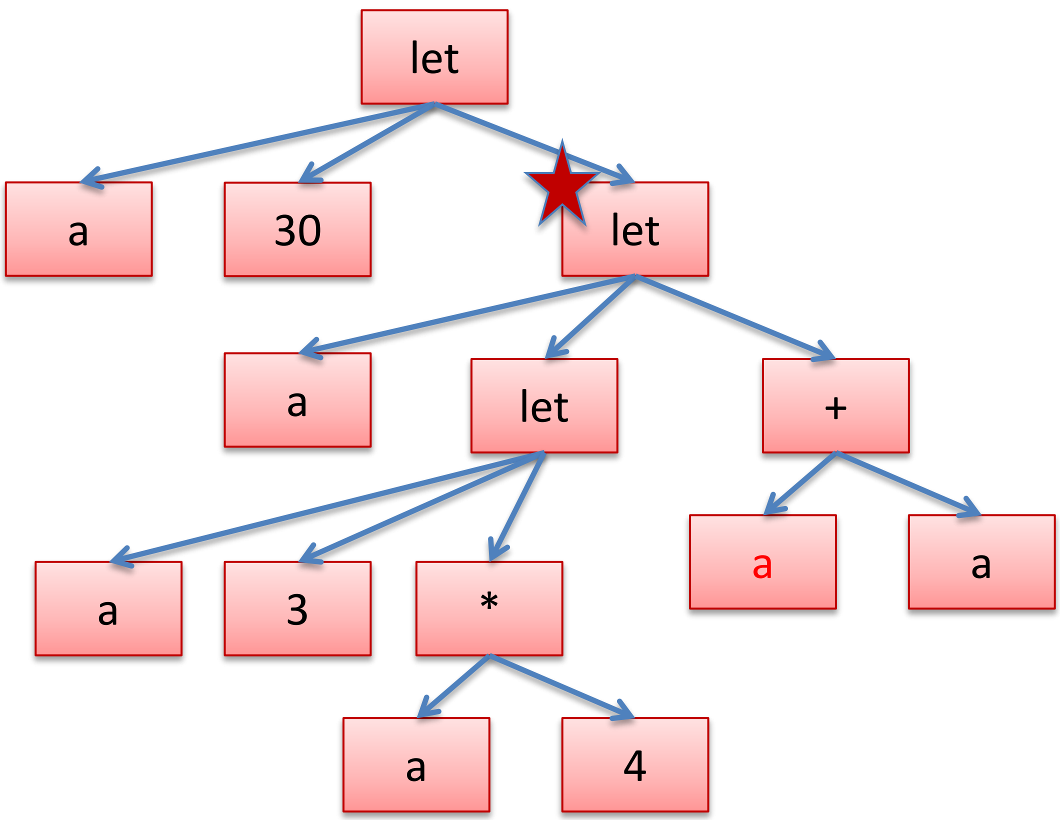
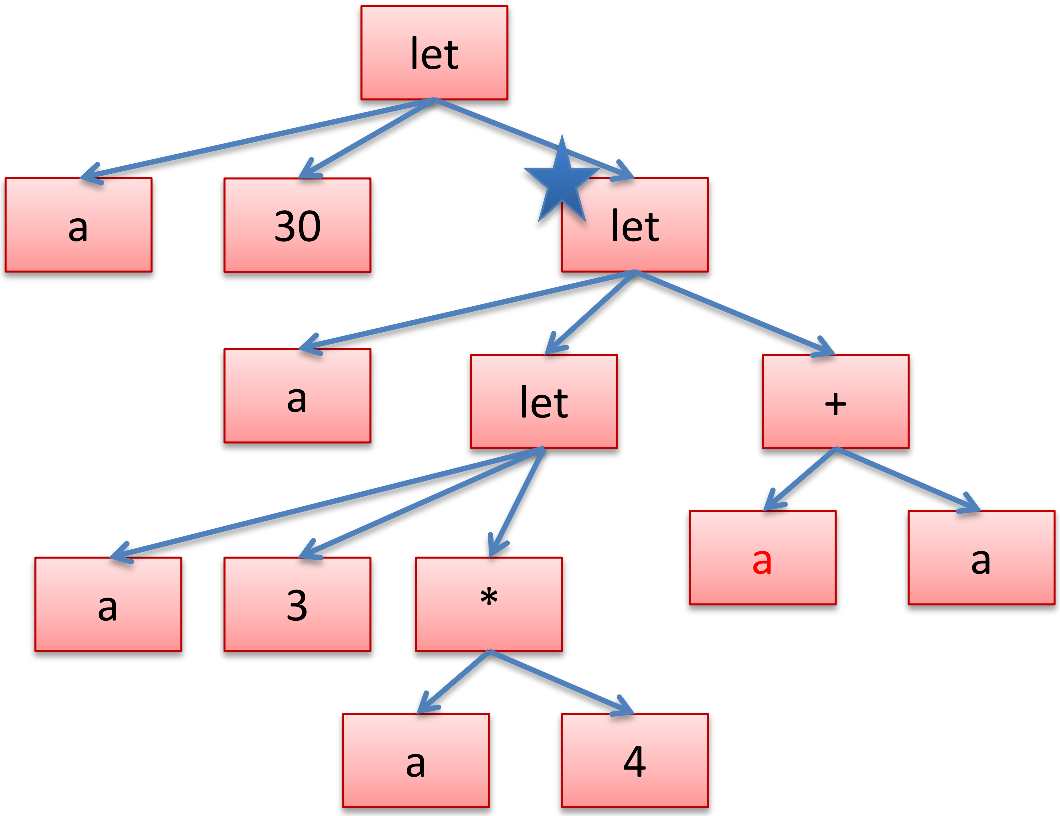


Abstract Syntax Trees

and checking if the “let” binds the variable – if so, we’ve found the nearest enclosing definition. If not, we keep going up.

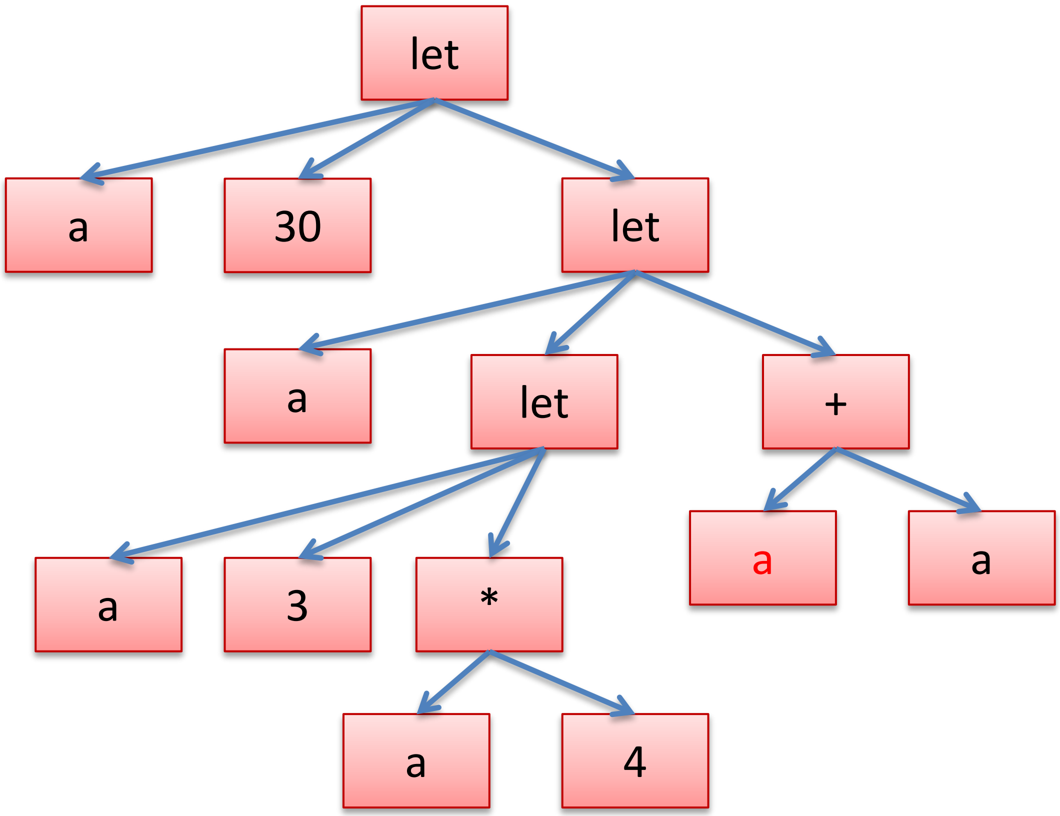
```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a
```



Abstract Syntax Trees

Now we can also systematically rename the variables so that it's not so confusing. Systematic renaming is called *alpha-conversion*

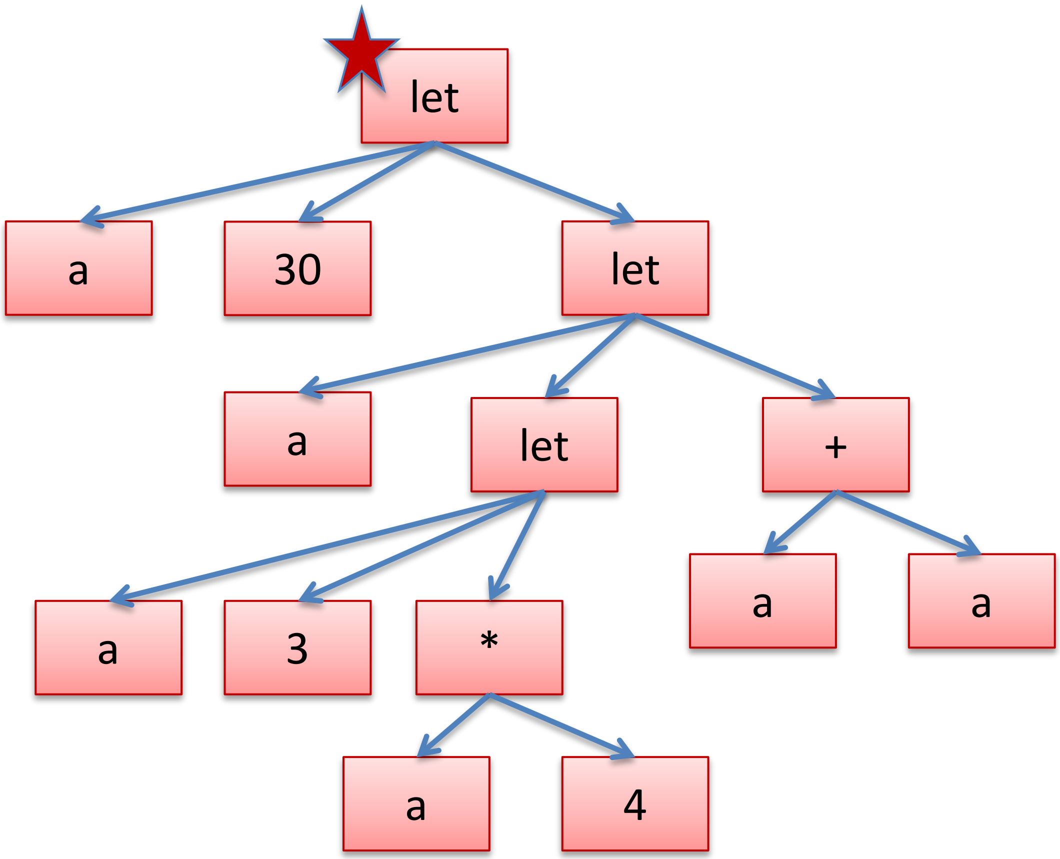
```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a
```



Abstract Syntax Trees

Start with a let, and pick a fresh variable name, say “x”

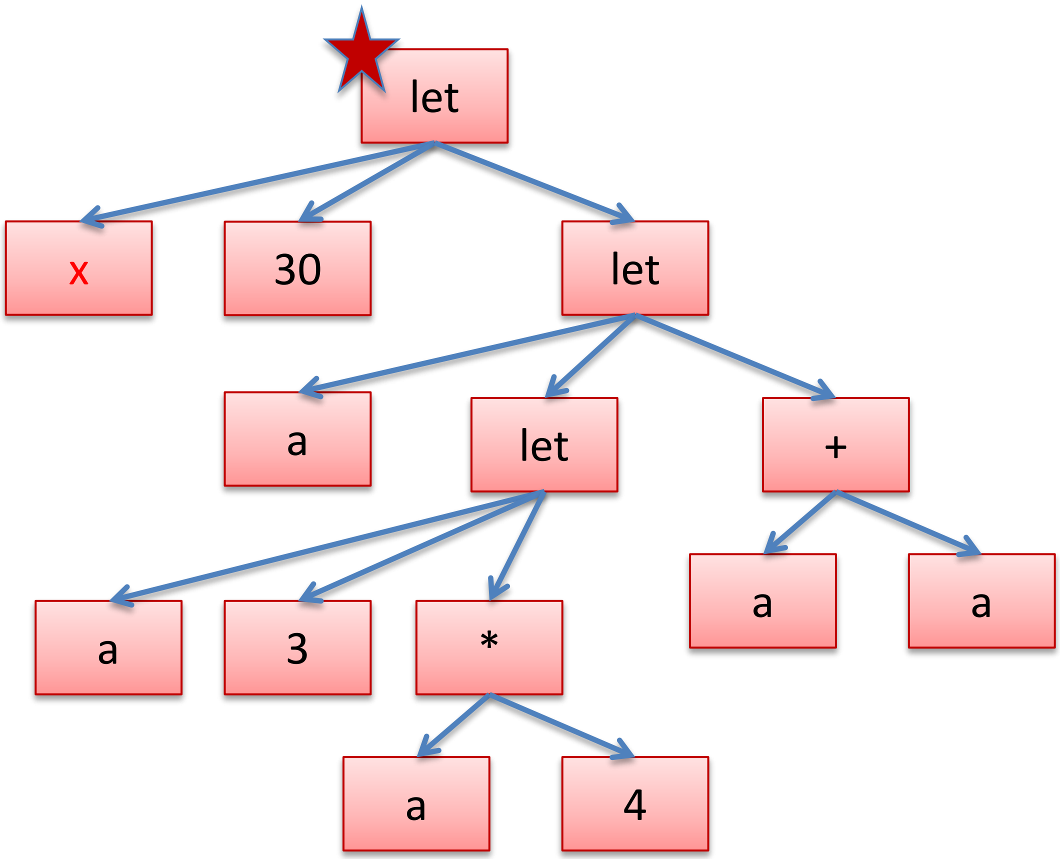
```
let a = 30 in
let a =
  (let a = 3 in a*4)
in
a+a
```



Abstract Syntax Trees

Rename the binding occurrence from “a” to “x”.

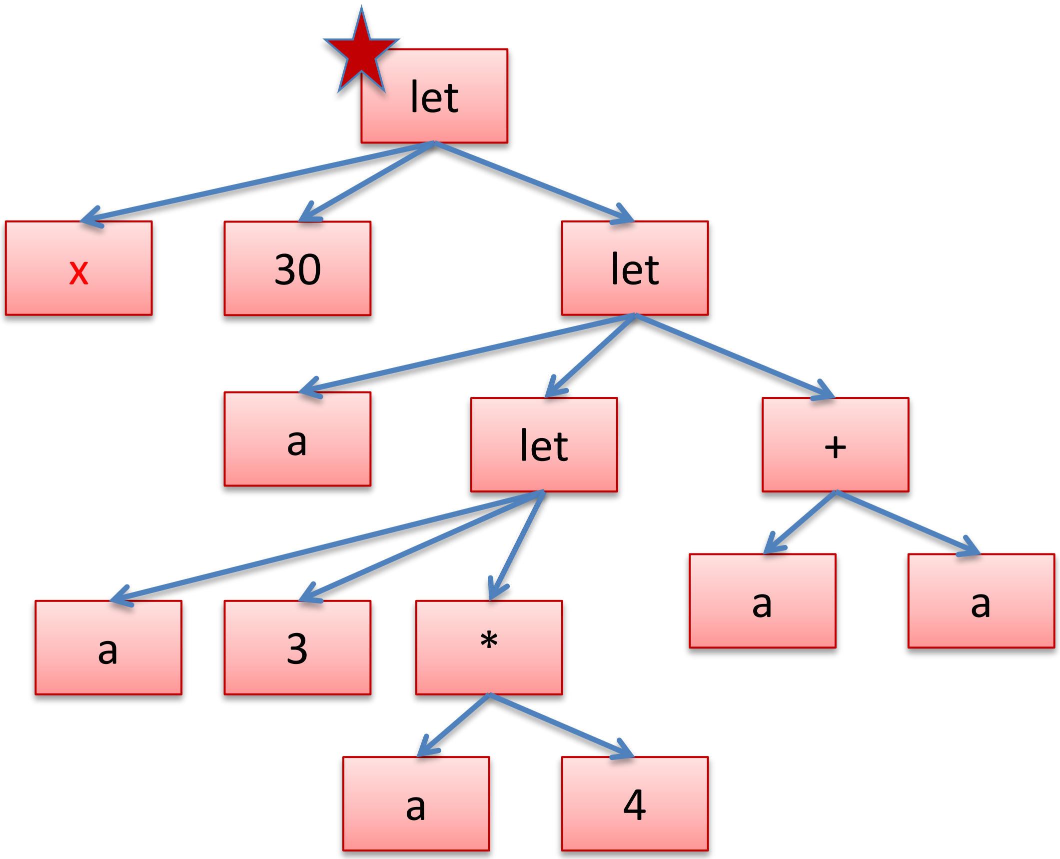
```
let x = 30 in
let a =
  (let a = 3 in a*4)
in
a+a
```



Abstract Syntax Trees

Then rename all of the occurrences of the variables ***that this let binds.***

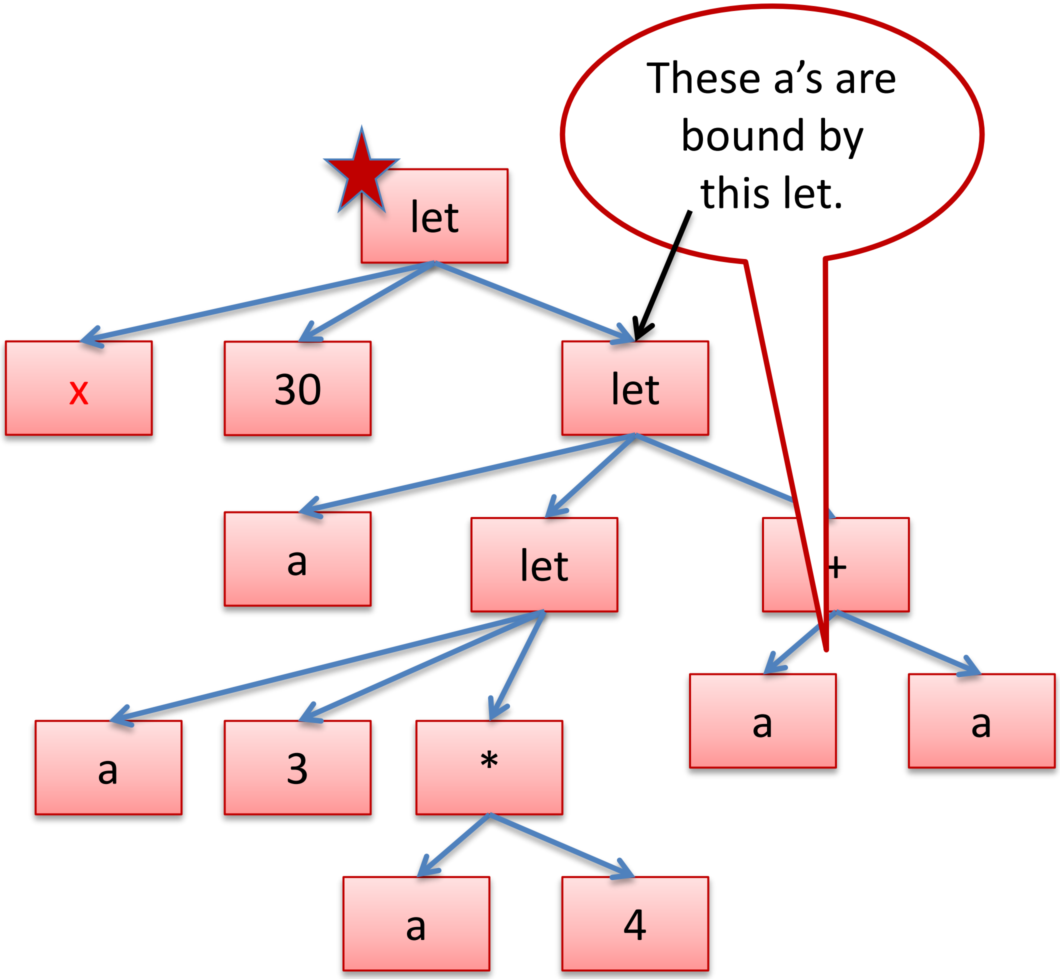
```
let x = 30 in
let a =
  (let a = 3 in a*4)
in
a+a
```



Abstract Syntax Trees

There are none in this case!

```
let x = 30 in
let a =
  (let a = 3 in a*4)
in
a+a
```



Abstract Syntax Trees

There are none in this case!

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let x = 30 in
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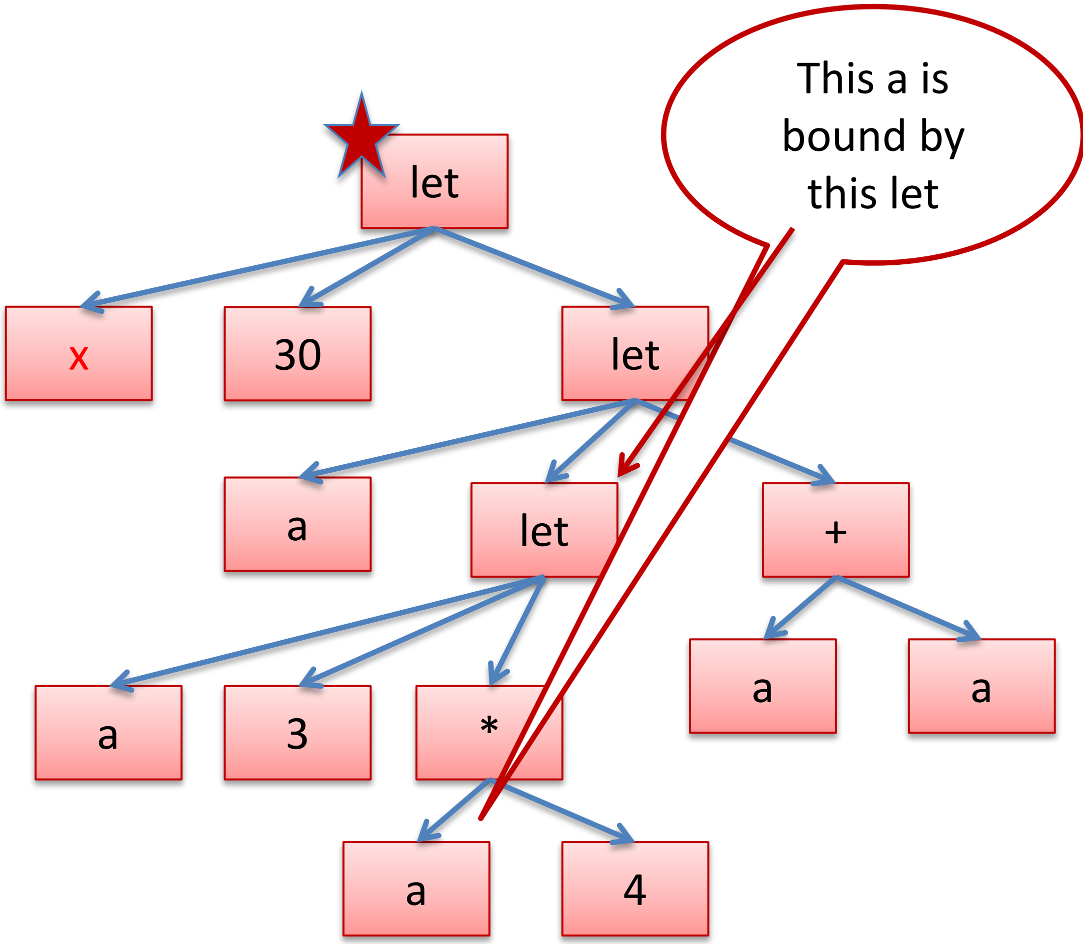
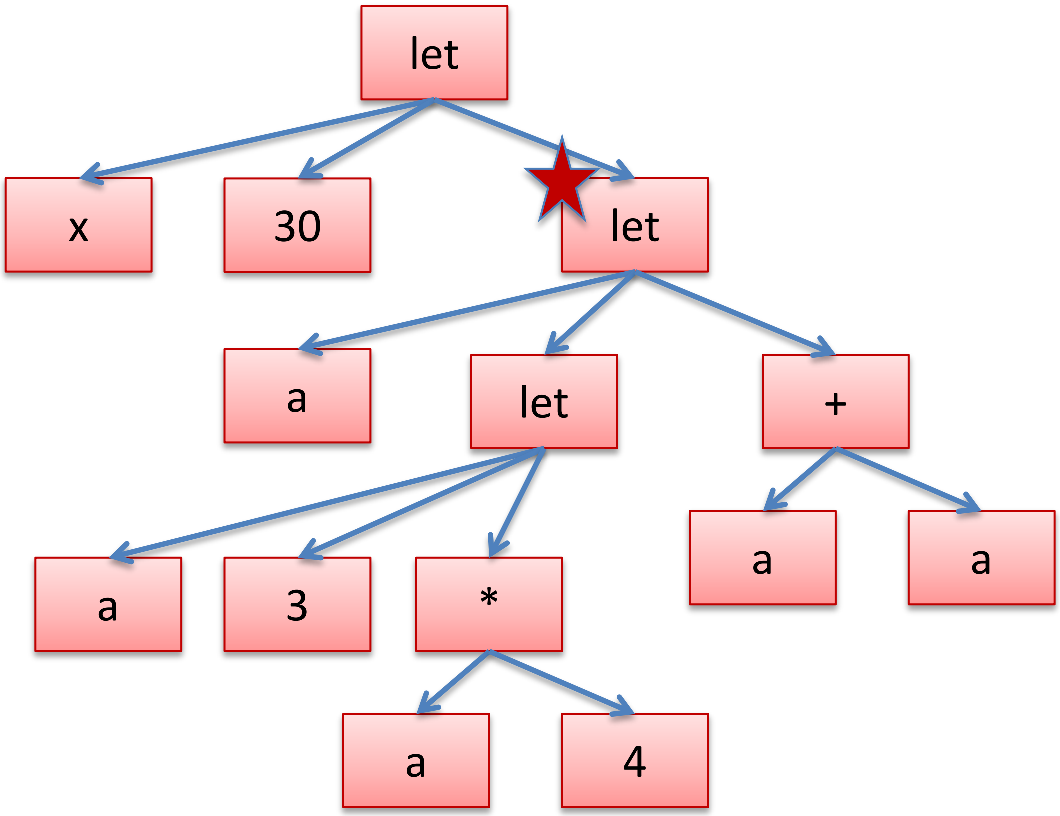


Abstract Syntax Trees

Let’s do another let, renaming “a” to “y”.

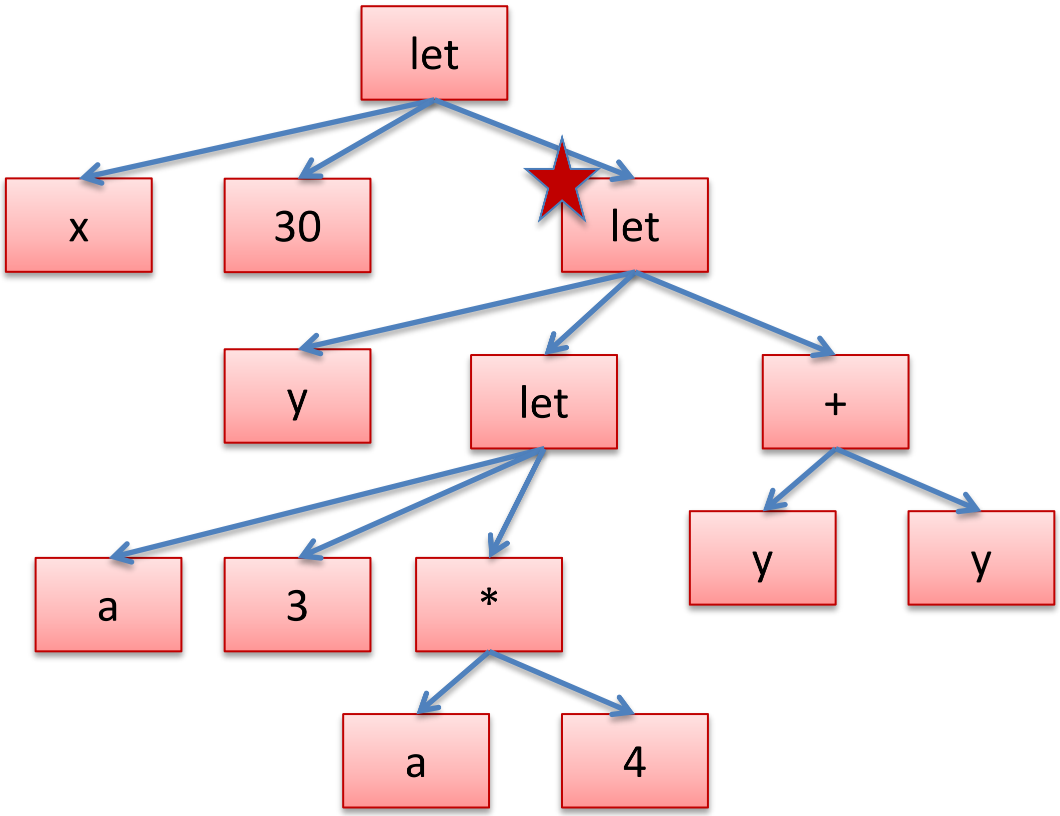
```
let x = 30 in
let a =
  (let a = 3 in a*4)
in
a+a
```



Abstract Syntax Trees

Let’s do another let, renaming “a” to “y”.

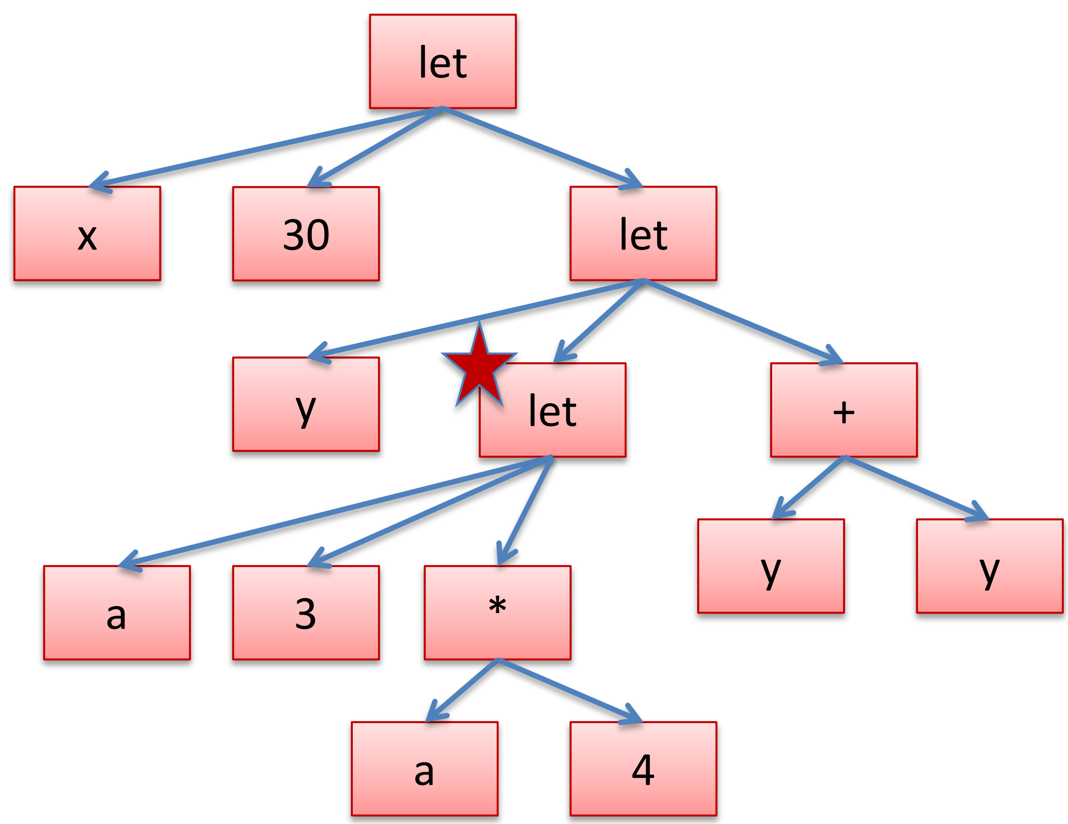
```
let x = 30 in
let y =
  (let a = 3 in a*4)
in
y+y
```



Abstract Syntax Trees

And if we rename the other let to "z":

```
let x = 30 in
let y =
  (let z = 3 in z*4)
in
y+y
```



Implementing Renaming


```
type var = string
type op = Plus | Minus
type exp =
   | Int_e of int
   | Op_e  of exp * op * exp
   | Var_e of var
   | Let_e of var * exp * exp
```

```
let rec rename (x:var) (y:var) (e:exp) : exp =
```

Implementing Renaming


```
type var = string
type op = Plus | Minus
type exp =
   | Int_e of int
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   | Var_e of var
   | Let_e of var * exp * exp
```

```
let rec rename (x:var) (y:var) (e:exp) : exp =
  match e with
  | Op_e  (e1, op, e2) ->

  | Var_e z ->

  | Int_e i ->

  | Let_e (z,e1,e2) ->
```

Implementing Renaming


```
type var = string
type op = Plus | Minus
type exp =
   | Int_e of int
   | Op_e  of exp * op * exp
   | Var_e of var
   | Let_e of var * exp * exp
```

```
let rec rename (x:var) (y:var) (e:exp) : exp =
  match e with
  | Op_e  (e1, op, e2) ->
      Op_e (rename x y e1, op, rename x y e2)
  | Var_e z ->

  | Int_e i ->

  | Let_e (z,e1,e2) ->
```

Implementing Renaming

```
type var = string
type op = Plus | Minus
type exp =
   | Int_e of int
   | Op_e  of exp * op * exp
   | Var_e of var
   | Let_e of var * exp * exp
```

```
let rec rename (x:var) (y:var) (e:exp) : exp =
  match e with
  | Op_e  (e1, op, e2) ->
      Op_e (rename x y e1, op, rename x y e2)
  | Var_e z ->
      if z = x then Var_e y else e
  | Int_e i ->

  | Let_e (z,e1,e2) ->
```

Implementing Renaming


```
type var = string
type op = Plus | Minus
type exp =
   | Int_e of int
   | Op_e  of exp * op * exp
   | Var_e of var
   | Let_e of var * exp * exp
```

```
let rec rename (x:var) (y:var) (e:exp) : exp =
  match e with
  | Op_e  (e1, op, e2) ->
      Op_e (rename x y e1, op, rename x y e2)
  | Var_e z ->
      if z = x then Var_e y else e
  | Int_e i ->
      Int_e i
  | Let_e (z,e1,e2) ->
```

Implementing Renaming


```
type var = string
type op = Plus | Minus
type exp =
   | Int_e of int
   | Op_e  of exp * op * exp
   | Var_e of var
   | Let_e of var * exp * exp
```

```
let rec rename (x:var) (y:var) (e:exp) : exp =
  match e with
  | Op_e  (e1, op, e2) ->
      Op_e (rename x y e1, op, rename x y e2)
  | Var_e z ->
      if z = x then Var_e y else e
  | Int_e i ->
      Int_e i
  | Let_e (z,e1,e2) ->
      Let_e (z, rename x y e1,
                if z = x then e2 else rename x y e2)
```

Evaluation

recall, we write:

```
e1  -->  e2
```

to indicate that e1 evaluates to e2 in a single step

for example:

```
2 + 3  -->  5
```

Evaluation


```
let x = 30 in
let y = 20 + x in
x+y
```

Evaluation


```
let x = 30 in
let y = 20 + x in
x+y
```

-->

```
let y = 20 + 30 in
30+y
```

Notice: we do a step of evaluation by *substituting* the value 30 for all the uses of x

Evaluation


```
let x = 30 in
let y = 20 + x in
x+y
```

-->

```
let y = 20 + 30 in
30+y
```

-->

```
let y = 50 in
30+y
```

In this step, we just evaluated the right-hand side of the let. We now have a *value* (50) on the right-hand side.

Evaluation


```
let x = 30 in
let y = 20 + x in
x+y
```

-->

```
let y = 20 + 30 in
30+y
```

-->

```
let y = 50 in
30+y
```

-->

```
30+50
```

substitution again

Evaluation


```
let x = 30 in
let y = 20 + x in
x+y
```

-->

```
let y = 20 + 30 in
30+y
```

-->

```
let y = 50 in
30+y
```

-->

```
30+50
```

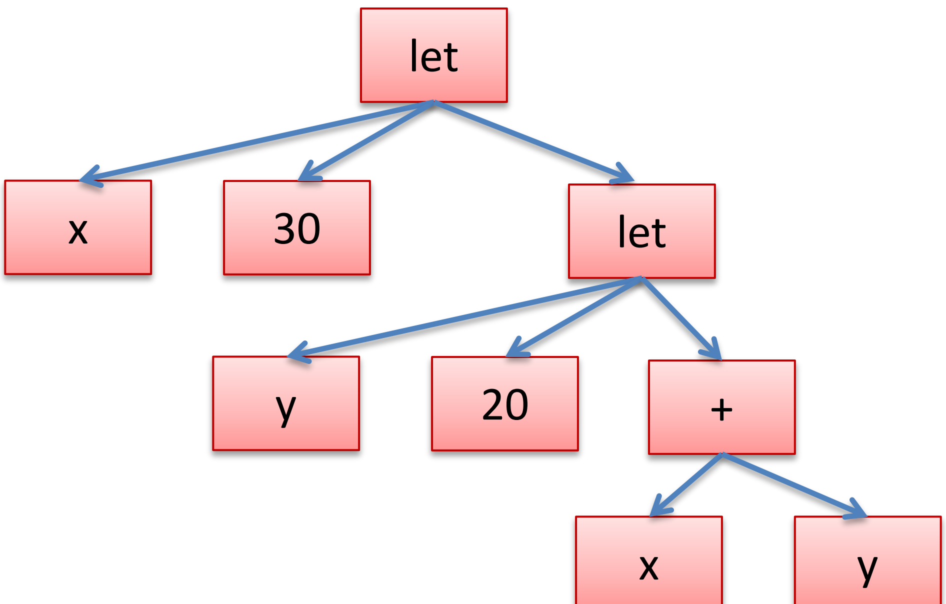
-->

```
80
```

evaluation complete: we have produced a *value*

Evaluation


```
let x = 30 in
let y = 20 in
x+y
```

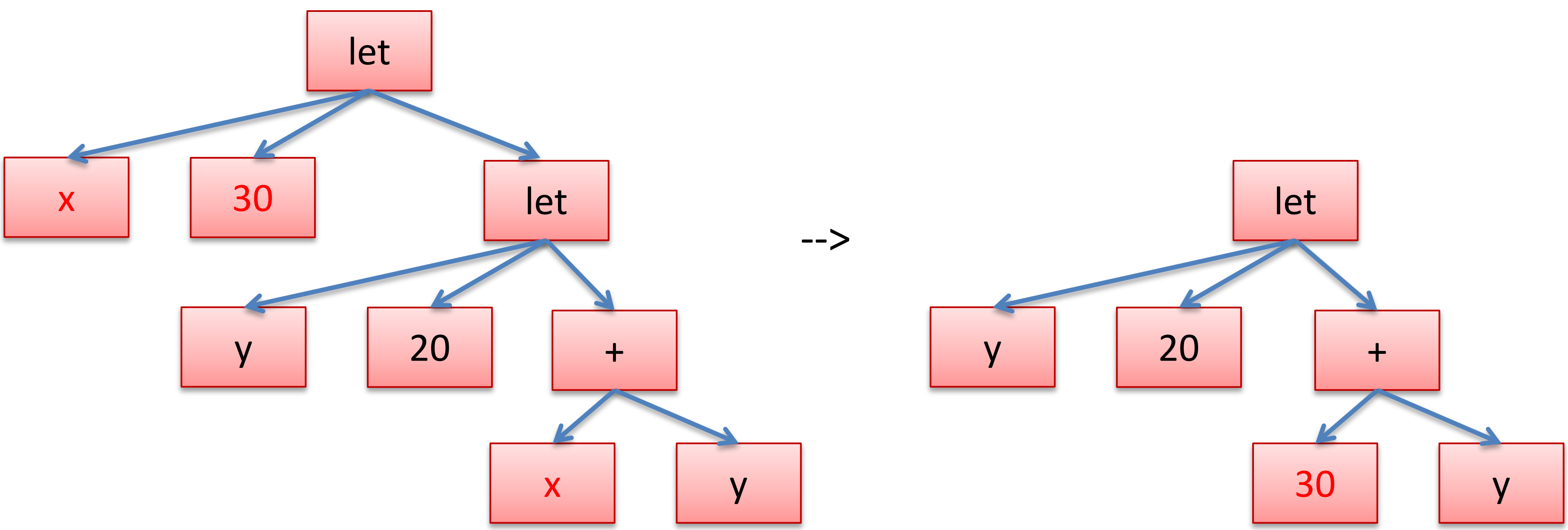


Evaluation via Substitution

```
let x = 30 in
let y = 20 in
x+y
```

-->

```
let y = 20 in
30+y
```



Binding occurrences versus applied occurrences


```
type exp =
  | Int_e of int
  | Op_e of exp * op * exp
  | Var_e of variable
  | Let_e of variable * exp * exp
```

This is a use of a variable

This is a binding occurrence of a variable

A Useful Auxiliary Function

nested “|” pattern
(can’t use variables)

```
let is_value (e:exp) : bool =
  match e with
  | Int_e _ -> true
  | ( Op_e  _
    | Let_e _
    | Var_e _ ) -> false
```

Recall: A *value* is a successful result of a computation.
Once we have computed a value, there is no more work to be done.

Integers (3), strings ("hi"), functions ("fun x -> x + 2") are values.

Operations ("x + 2"), function calls ("f x"), match statements are not value.

Two Other Auxiliary Functions


```
(* eval_op v1 o v2:
   apply o to v1 and v2 *)
eval_op    : value -> op -> value -> exp

(* substitute v x e:
   replace free occurrences of x with v in e *)
substitute : value -> variable -> exp -> exp
```

A Simple Evaluator


```
is_value   : exp -> bool
eval_op    : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp = ...


(* Goal:  evaluate e; return resulting value *)
```

A Simple Evaluator

```
is_value   : exp -> bool
eval_op    : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
  | Int_e i ->
  | Op_e(e1,op,e2) ->



  | Let_e(x,e1,e2) ->
```

A Simple Evaluator


```
is_value   : exp -> bool
eval_op    : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) ->




  | Let_e(x,e1,e2) ->
```

A Simple Evaluator


```
is_value   : exp -> bool
eval_op    : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) ->
         let v1 = eval e1 in
         let v2 = eval e2 in
         eval_op v1 op v2
  | Let_e(x,e1,e2) ->
```

A Simple Evaluator


```
is_value   : exp -> bool
eval_op    : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) ->
         let v1 = eval e1 in
         let v2 = eval e2 in
         eval_op v1 op v2
  | Let_e(x,e1,e2) ->
         let v1  = eval e1 in
         let e2' = substitute v1 x e2 in
         eval e2'
```

Shorter but Dangerous


```
is_value    : exp -> bool
eval_op     : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) ->
        eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) ->
        eval (substitute (eval e1) x e2)
```

Why?

Simpler but Dangerous


```
is_value    : exp -> bool
eval_op     : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) ->
        eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) ->
        eval (substitute (eval e1) x e2)
```

Which gets evaluated first?
Does OCaml use left-to-right eval order or right-to-left?
Always use OCaml let if you want to specify evaluation order.

Simpler but Dangerous


```
is_value   : exp -> bool
eval_op    : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) ->
       eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) ->
       eval (substitute (eval e1) x e2)
```

Since the language we are interpreting is *pure* (no effects),
it won't matter which expression gets evaluated first.
We'll produce the same answer in either case.

Simpler but Dangerous


```
is_value   : exp -> bool
eval_op    : value -> op -> value -> value
substitute : value -> variable -> exp -> exp

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) ->
        eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) ->
        eval (substitute (eval e1) x e2)
```

Quick question:
Do you notice anything else suspicious here about this code?
Anything OCaml might flag?

Oops! We Missed a Case:


```
let eval_op v1 op v2 = …
let substitute v x e = …

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> ???
```

If we start out with an expression with no *free variables*, we will never run into a free variable when we evaluate.
Every variable gets replaced by a value as we compute, via substitution.

Theorem: Well-typed programs have no free variables.

We could leave out the case for variables, but that will create a mess of OCaml warnings – bad style. (Bad for debugging.)

We Could Use Options:


```
let eval_op v1 op v2 = …
let substitute v x e = …

let rec eval (e:exp) : exp option =
  match e with
  | Int_e i -> Some(Int_e i)
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> None
```

But this isn't quite right – we need to match on the recursive calls to eval to make sure we get Some value!

Exceptions


```
exception UnboundVariable of variable

let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x)
```

Instead, we can throw an exception.

Exceptions


```
exception UnboundVariable of variable

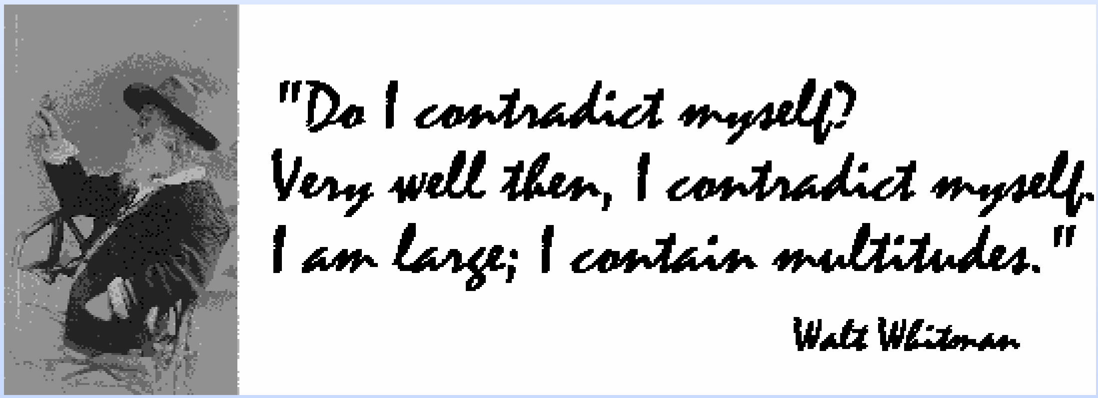
let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x)
```

Note that an exception declaration is a lot like a datatype declaration. Really, we are extending one big datatype (exn) with a new constructor (UnboundVariable).

Later on, we'll see how to catch an exception.

Exception or Option?

In a previous lecture, I railed against Java for all of the null pointer exceptions it raised. Should we use options or exns?



There are some rules; there is some taste involved.

- For errors/circumstances that *will occur*, use options
  - e.g.: the input might be ill formatted
- For errors that *cannot occur* (unless the program itself has a bug) and for which there are few "entry points" (few places checks needed) use exceptions
  - Java does not follow this rule: objects may be null *everywhere*

AUXILIARY FUNCTIONS

Evaluating the Primitive Operations


```
let eval_op (v1:exp) (op:operand) (v2:exp) : exp =
  match v1, op, v2 with
  | Int_e i, Plus,  Int_e j -> Int_e (i+j)
  | Int_e i, Minus, Int_e j -> Int_e (i-j)
  | Int_e i, Times, Int_e j -> Int_e (i*j)
  | _ ,(Plus | Minus | Times), _ ->
    if is_value v1 && is_value v2 then raise TypeError
    else raise NotValue
```

```
let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x)
```

Substitution

Want to replace x
(and only x) with v.

```
let substitute (v:exp) (x:variable) (e:exp) : exp =

    ...
```

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ ->
         | Op_e(e1,op,e2) ->
         | Var_e y ->               ... use x ...
         | Let_e (y,e1,e2) ->       ... use x ...



  in
  subst e
```

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) ->
         | Var_e y ->
         | Let_e (y,e1,e2) ->




  in
  subst e
```

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
        match e with
        | Int_e _ -> e
        | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
        | Var_e y ->
        | Let_e (y,e1,e2) ->



  in
  subst e
```

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
         | Var_e y -> if x = y then v else e
         | Let_e (y,e1,e2) ->




  in
  subst e
```

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
         | Var_e y -> if x = y then v else e
         | Let_e (y,e1,e2) ->
              Let_e (y,
                     subst e1,
                     subst e2)
  in
  subst e
```

WRONG!

Substitution

```
  let substitute (v:exp) (x:variable) (e:exp) : exp =
    let rec subst (e:exp) : exp =
           match e with
           | Int_e _ -> e
           | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
           | Var_e y -> if x = y then v else e
           | Let_e (y,e1,e2) ->
                Let_e (y,
                       if x = y then e1 else subst e1,
                       if x = y then e2 else subst e2)
    in
    subst e
```

wrong

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
         | Var_e y -> if x = y then v else e
         | Let_e (y,e1,e2) ->
              Let_e (y,
                     subst e1,
                     if x = y then e2 else subst e2)
  in
  subst e
;;
```


evaluation/substitution must implement our variable *scoping* rules correctly

Substitution

```
let substitute (v:exp) (x:variable) (e:exp) : exp =
  let rec subst (e:exp) : exp =
         match e with
         | Int_e _ -> e
         | Op_e(e1,op,e2) -> Op_e(subst e1,op,subst e2)
         | Var_e y -> if x = y then v else e
         | Let_e (y,e1,e2) ->
              Let_e (y,
                     subst e1,
                     if x = y then e2 else subst e2)
  in
  subst e
;;
```

If x and y are the same variable, then y *shadows* x.

SCALING UP THE LANGUAGE

(MORE FEATURES, MORE FUN)

Scaling up the Language


```
type exp = Int_e of int | Op_e of exp * op * exp
  | Var_e of variable | Let_e of variable * exp * exp
  | Fun_e of variable * exp | FunCall_e of exp * exp
```

Scaling up the Language


```
type exp = Int_e of int | Op_e of exp * op * exp
  | Var_e of variable | Let_e of variable * exp * exp
  | Fun_e of variable * exp | FunCall_e of exp * exp
```

OCaml's
fun x -> e
is represented as
Fun_e(x,e)

Scaling up the Language


```
type exp = Int_e of int | Op_e of exp * op * exp
  | Var_e of variable | Let_e of variable * exp * exp
  | Fun_e of variable * exp | FunCall_e of exp * exp
```

A function call
fact 3
is implemented as
FunCall_e (Var_e “fact”, Int_e 3)

Scaling up the Language


```
type exp = Int_e of int | Op_e of exp * op * exp
  | Var_e of variable | Let_e of variable * exp * exp
  | Fun_e of variable * exp | FunCall_e of exp * exp

let is_value (e:exp) : bool =
  match e with
  | Int_e _ -> true
  | Fun_e (_,_) -> true
  | (  Op_e (_,_,_)
     | Let_e (_,_,_)
     | Var_e _
     | FunCall_e (_,_) ) -> false
```

Functions are values!

Easy exam question:
What value does the OCaml interpreter produce when you enter (fun x -> 3) in to the prompt?
Answer: the value produced is (fun x -> 3)

Scaling up the Language


```
type exp = Int_e of int | Op_e of exp * op * exp
  | Var_e of variable | Let_e of variable * exp * exp
  | Fun_e of variable * exp | FunCall_e of exp * exp;;

let is_value (e:exp) : bool =
  match e with
  | Int_e _ -> true
  | Fun_e (_,_) -> true
  | (  Op_e (_,_,_)
     | Let_e (_,_,_)
     | Var_e _
     | FunCall_e (_,_) ) -> false
```

Function calls are not values.

Scaling up the Language


```
let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x)
  | Fun_e (x,e) -> Fun_e (x,e)
  | FunCall_e (e1,e2) ->
      (match eval e1, eval e2 with
       | Fun_e (x,e), v2 -> eval (substitute v2 x e)
       | _ -> raise TypeError)
```

Scaling up the Language

```
let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x)
  | Fun_e (x,e) -> Fun_e (x,e)
  | FunCall_e (e1,e2) ->
      (match eval e1, eval e2 with
       | Fun_e (x,e), v2 -> eval (substitute v2 x e)
       | _ -> raise TypeError)
```

values (including functions) always evaluate to themselves.

Scaling up the Language


```
let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x)
  | Fun_e (x,e) -> Fun_e (x,e)
  | FunCall_e (e1,e2) ->
      (match eval e1, eval e2 with
       | Fun_e (x,e), v2 -> eval (substitute v2 x e)
       | _ -> raise TypeError)
```

To evaluate a function call, we first evaluate both e1 and e2 to values.

Scaling up the Language


```
let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x)
  | Fun_e (x,e) -> Fun_e (x,e)
  | FunCall_e (e1,e2) ->
      (match eval e1, eval e2 with
       | Fun_e (x,e), v2 -> eval (substitute v2 x e)
       | _ -> raise TypeError)
```

e1 had better evaluate to a function value, else we have a type error.

Scaling up the Language


```
let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x)
  | Fun_e (x,e) -> Fun_e (x,e)
  | FunCall_e (e1,e2) ->
      (match eval e1, eval e2 with
       | Fun_e (x,e), v2 -> eval (substitute v2 x e)
       | _ -> raise TypeError)
```

Then we substitute e2's value (v2) for x in e and evaluate the resulting expression.

Simplifying a little

```
let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x)
  | Fun_e (x,e) -> Fun_e (x,e)
  | FunCall_e (e1,e2) ->
      (match eval e1 with
       | Fun_e (x,e) -> eval (substitute (eval e2) x e)
       | _ -> raise TypeError)
```

We don't really need
to pattern-match on e2.
Just evaluate here

Simplifying a little


```
let rec eval (e:exp) : exp =
  match e with
   | Int_e i -> Int_e i
   | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
   | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
   | Var_e x -> raise (UnboundVariable x)
   | Fun_e (x,e) -> Fun_e (x,e)
   | FunCall_e (ef,e1) ->
       (match eval ef with
        | Fun_e (x,e2) -> eval (substitute (eval e1) x e2)
        | _ -> raise TypeError)
```

This looks like
the case for let!

Let and Lambda


```
let x = 1 in x+41
-->
1+41
-->
42
```

```
(fun x -> x+41) 1
-->
1+41
-->
42
```

In general:

```
(fun x -> e2) e1    ==   let x = e1 in e2
```

So we could write:

```
let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (FunCall (Fun_e (x,e2), e1))
  | Var_e x -> raise (UnboundVariable x)
  | Fun_e (x,e) -> Fun_e (x,e)
  | FunCall_e (ef,e2) ->
      (match eval ef with
       | Fun_e (x,e1) -> eval (substitute (eval e1) x e2)
       | _ -> raise TypeError)
```

In programming-languages speak: "Let is *syntactic sugar* for a function call"

Syntactic sugar: A new feature defined by a simple, local transformation.

Recursive definitions


```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp |
   | Fun_e of variable * exp | FunCall_e of exp * exp
   | Rec_e of variable * variable * exp
```

```
let rec f x = f (x+1) in f 3
```

(rewrite)

```
let f = (rec f x -> f (x+1)) in
f 3
```

(alpha-convert)

```
let g = (rec f x -> f (x+1)) in
g 3
```

(implement)

```
Let_e (“g,
  Rec_e (“f”, “x”,
    FunCall_e (Var_e “f”, Op_e (Var_e “x”, Plus, Int_e 1))
 ),
 FunCall (Var_e “g”, Int_e 3)
)
```

Recursive definitions


```
type exp = Int_e of int | Op_e of exp * op * exp
   | Var_e of variable | Let_e of variable * exp * exp |
   | Fun_e of variable * exp | FunCall_e of exp * exp
   | Rec_e of variable * variable * exp
```

```
let is_value (e:exp) : bool =
  match e with
   | Int_e _ -> true
   | Fun_e (_,_) -> true
   | Rec_e of (_,_,_) -> true
   | (Op_e (_,_,_) | Let_e (_,_,_) |
     Var_e _ | FunCall_e (_,_) ) -> false
```

Recursive definitions


```
type exp = Int_e of int | Op_e of exp * op * exp
  | Var_e of variable | Let_e of variable * exp * exp |
  | Fun_e of variable * exp | FunCall_e of exp * exp
  | Rec_e of variable * variable * exp
```

```
let is_value (e:exp) : bool =
  match e with
  | Int_e _ -> true
  | Fun_e (_,_) -> true
  | Rec_e o  (_,_,_) -> true
  | (Op_e (_,   | Let_e (_,_,_) |
     Var_e _
```

Fun_e (x, body) == Rec_e("unused", x, body)

A better IR would just delete Fun_e – avoid unnecessary redundancy

Interlude: Notation for Substitution

"Substitute value v for variable x in expression e:" e [v / x]

examples of substitution:

(x + y) [7/y]	is	(x + 7)
(let x =30 in let y=40 in x + y) [7/y]	is	(let x =30 in let y=40 in x + y)
(let y = y in let y = y in y + y) [7/y]	is	(let y = 7 in let y = y in y + y)

Evaluating Recursive Functions

Basic evaluation rule for recursive functions:

```
(rec f x = body) arg      -->      body [arg/x] [rec f x = body/f]
```

argument value substituted for parameter

entire function substituted for function name

Evaluating Recursive Functions

Start out with
a let bound to
a recursive function:

```
let g =
  rec f x ->
    if x <= 0 then x
    else x + f (x-1)
in g 3
```

The Substitution:

```
g 3 [rec f x ->
       if x <= 0 then x
       else x + f (x-1)  / g]
```

The Result:

```
(rec f x ->
   if x <= 0 then x else x + f (x-1)) 3
```

Evaluating Recursive Functions

Recursive Function Call:

```
(rec f x ->
   if x <= 0 then x else x + f (x-1)) 3
```

The Substitution:

```
(if x <= 0 then x else x + f (x-1))
  [ rec f x ->
        if x <= 0 then x
        else x + f (x-1)  / f ]
  [ 3 / x ]
```

Substitute argument for parameter

Substitute entire function for function name

The Result:

```
(if 3 <= 0 then 3 else 3 +
   (rec f x ->
        if x <= 0 then x
        else x + f (x-1)) (3-1))
```

Evaluating Recursive Functions


```
let rec eval (e:exp) : exp =
  match e with
  | Int_e i -> Int_e i
  | Op_e(e1,op,e2) -> eval_op (eval e1) op (eval e2)
  | Let_e(x,e1,e2) -> eval (substitute (eval e1) x e2)
  | Var_e x -> raise (UnboundVariable x)
  | Fun_e (x,e) -> Fun_e (x,e)
  | FunCall_e (e1,e2) ->
      (match eval e1 with
       | Fun_e (x,e) ->
           let v = eval e2 in
           substitute e x v

       | (Rec_e (f,x,e)) as f_val ->
           let v = eval e2 in
           let body = substitute f_val f
                          (substitute v x e) in
           eval body

       | _ -> raise TypeError)
```

pattern as *x*

match the pattern and binds x to value

More Evaluation

```
(rec fact n = if n <= 1 then 1 else n * fact(n-1)) 3
-->
if 3 < 1 then 1 else
  3 * (rec fact n = if ... then ... else ...) (3-1)
-->
3 * (rec fact n = if … ) (3-1)
-->
3 * (rec fact n = if … ) 2
-->
3 * (if 2 <= 1 then 1 else 2 * (rec fact n = ...)(2-1))
-->
3 * (2 * (rec fact n = ...)(2-1))
-->
3 * (2 * (rec fact n = ...)(1))
-->
3 * 2 * (if 1 <= 1 then 1 else 1 * (rec fact ...)(1-1))
-->
3 * 2 * 1
```

Summary

Datatypes are very useful for *representing* the *abstract syntax* of programming languages

- Moral: If you are going to implement a programming language, you really should be using a functional language with data types

Interpreters are recursive programs that evaluate expressions and produce values.