# Lazy Evaluation & Infinite Data

COS 326

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Some ideas in this lecture borrowed from Brigitte Pientka, McGill University

# AN INFINITE DATA STRUCTURE: STREAMS

# Streams

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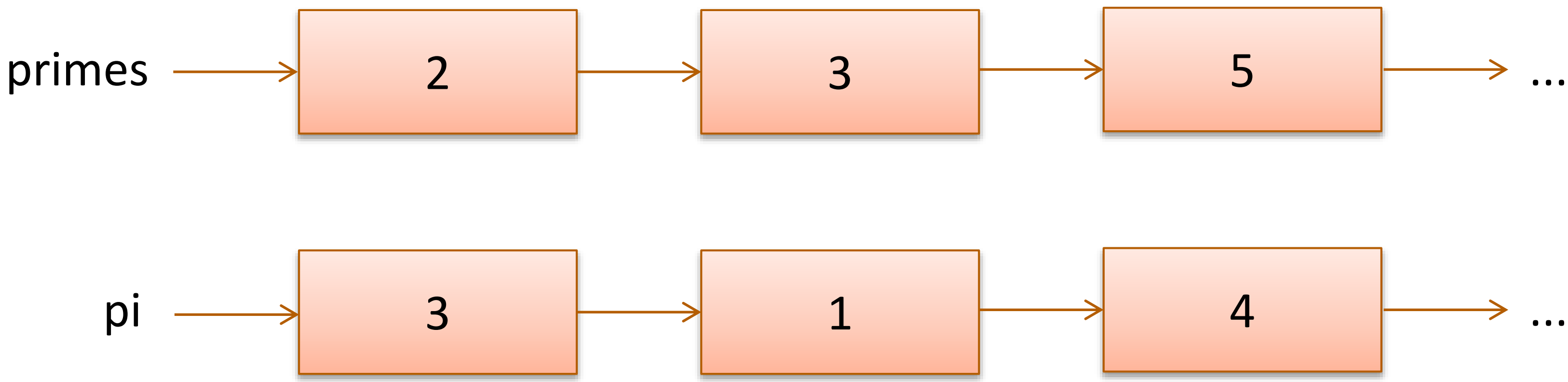
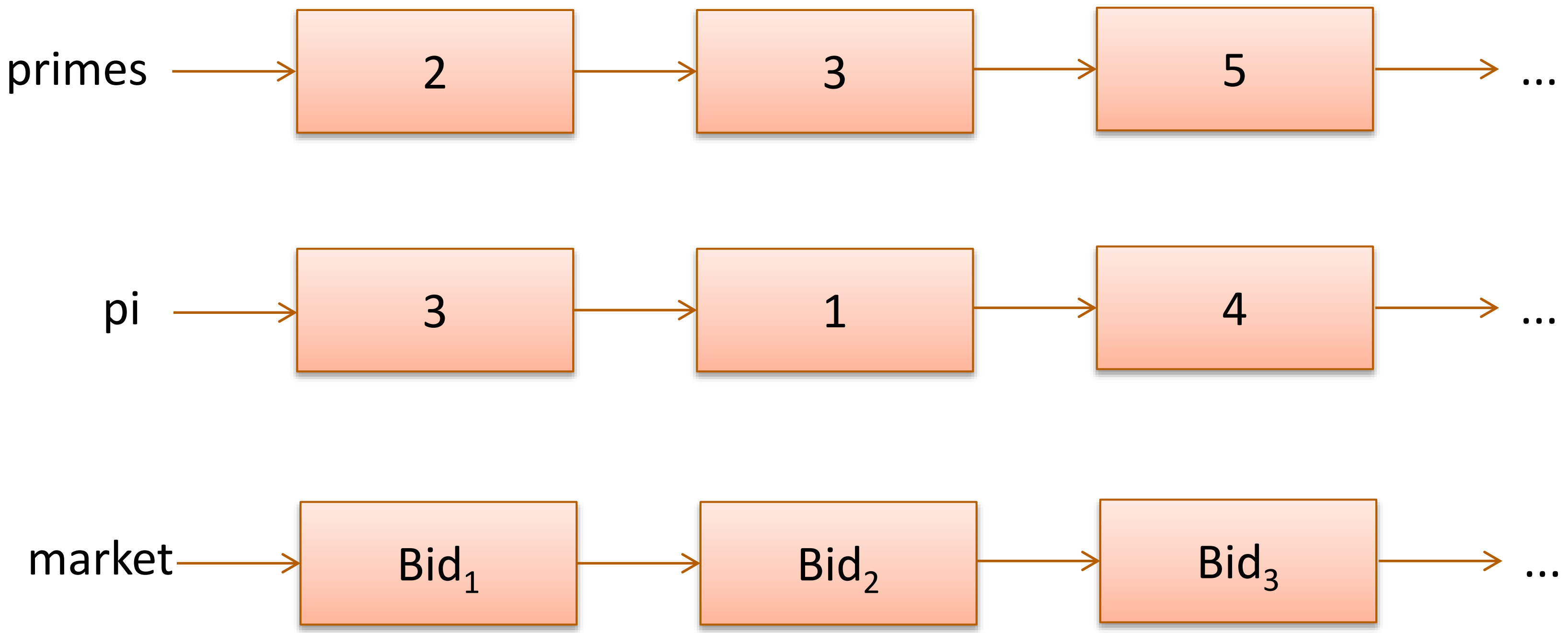


# Streams

Sometimes it is useful to define the entirety of an infinite data set *now* and sample finite parts of it *later* ...



# Consider this definition:

```
type 'a stream =
  Cons of 'a * ('a stream)
```

We can write functions to extract the head and tail of a stream:

```
let head(s:'a stream):'a =
  match s with
  | Cons (h,_) -> h

let tail(s:'a stream):'a stream =
  match s with
  | Cons (_,t) -> t
```

# But there's a problem…

```
type 'a stream =
  Cons of 'a * ('a stream)
```

How do I build a value of type 'a stream?

Cons (3, Cons (4, ___))

Cons (3, ___)

# But there's a problem...

```
type 'a stream =
  Cons of 'a * ('a stream)
```

How do I build a value of type 'a stream?

Cons (3, Cons (4, ___))

Cons (3, ___)

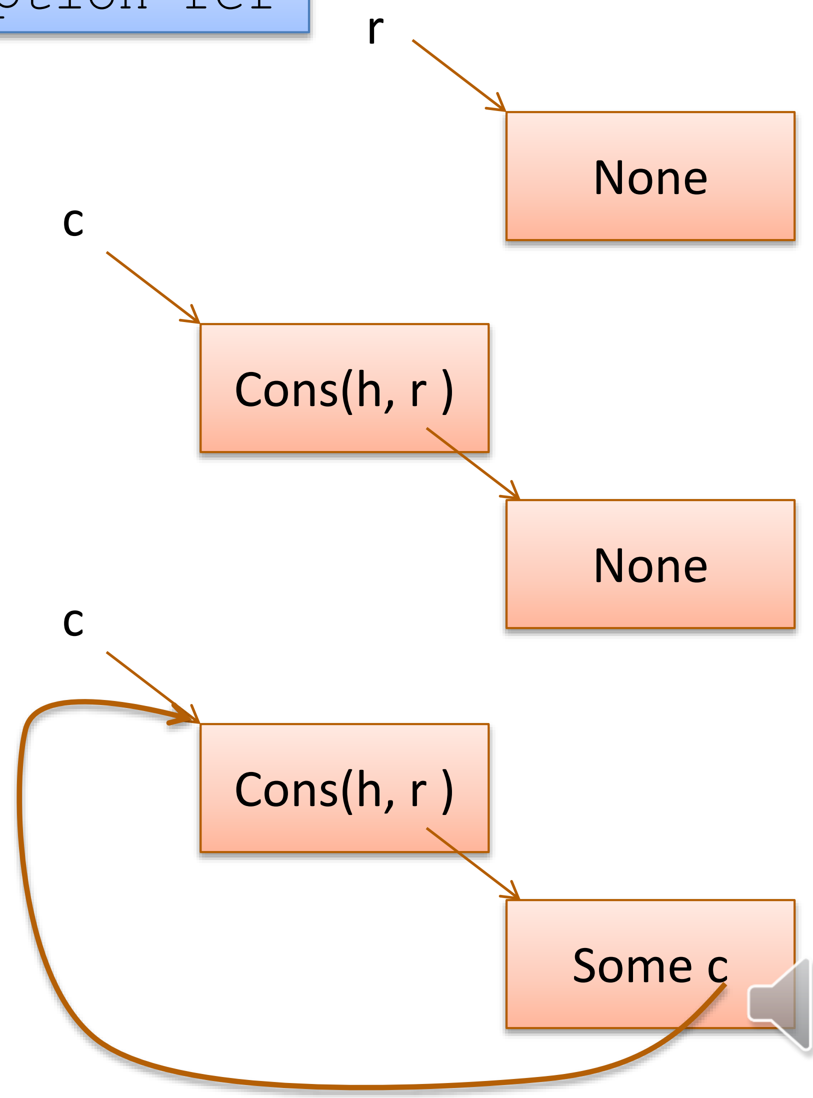
There doesn't seem to be a base case (e.g., Nil)

Since we need a stream to build a stream,
what can we do to get started?

# An alternative would be to use refs

```
type 'a stream =
  Cons of 'a * ('a stream) option ref
```

```
let circular_cons h =
  let r = ref None in
  let c = Cons(h,r) in
  (r := (Some c); c)
```




This works ...

but has a serious drawback

# An alternative would be to use refs

```
type 'a stream =
  Cons of 'a * ('a stream) option ref
```

```
let circular_cons h =
  let r = ref None in
  let c = Cons(h,r) in
  (r := (Some c); c)
```

.... when we try to get out the tail, it may not exist.

# Back to our earlier idea

```
type 'a stream =
  Cons of 'a * ('a stream)
```

Let's look at creating the stream of all natural numbers:

```
let rec nats i = Cons(i,nats (i+1))
```

```
# let n = nats 0;;
Stack overflow during evaluation (looping recursion?).
```

OCaml evaluates our code just a little bit too *eagerly*.
We want to evaluate the right-hand side *only when necessary* ...

# Another idea

One way to implement "waiting" is to wrap a computation up in a function and then call that function later when we want to.

Another attempt:

```
type 'a stream =
  Cons of 'a * ('a stream)
```

```
let rec ones =
  fun () -> Cons(1,ones)
```

```
let head x =
  match x () with
    Cons (hd, tail) -> hd
```

Are there any problems with this code?

Darn. Doesn't type check!
It's a function with type
unit -> int stream
not just int stream

# Functional Implementation

What if we changed the definition of streams one more time?

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

```
let rec ones : int stream =
  fun () -> Cons(1,ones)
```

*mutually recursive* type definition

Or, the way we'd normally write it:

```
let rec ones () = Cons(1,ones)
```

# Functional Implementation

How would we define head, tail, and map of an 'a stream?

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

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```
let head(s:'a stream):'a =
...
```

# Functional Implementation

How would we define head, tail, and map of an ’a stream?

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type 'a str = Cons of 'a * ('a stream)
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let head(s:'a stream):'a =
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```

# Functional Implementation

How would we define head, tail, and map of an ’a stream?

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type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

```
let head(s:'a stream):'a =
 match s() with
 | Cons(h,_) -> h
```

```
let tail(s:'a stream):'a stream =
  ...
```

# Functional Implementation

How would we define head, tail, and map of an ’a stream?

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

```
let head(s:'a stream):'a =
 match s() with
 | Cons(h,_) -> h
```

```
let tail(s:'a stream):'a stream =
 match s() with
 | Cons(_,t) -> t
```

# Functional Implementation

How would we define head, tail, and map of an ’a stream?

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

```
let rec map (f:'a->'b) (s:'a stream) : 'b stream =
...
```

# Functional Implementation

How would we define head, tail, and map of an ’a stream?

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

```
let rec map (f:'a->'b) (s:'a stream) : 'b stream =
  Cons(f (head s), map f (tail s))
```

# Functional Implementation

How would we define head, tail, and map of an ’a stream?

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

```
let rec map (f:'a->'b) (s:'a stream) : 'b stream =
  Cons(f (head s), map f (tail s))
```

Rats!

Infinite looping!

# Functional Implementation

How would we define head, tail, and map of an ’a stream?

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

```
let rec map (f:'a->'b) (s:'a stream) : 'b stream =
  Cons(f (head s), map f (tail s))
```

Doesn’t type check!
Cons (x,y) is a **str** not a **stream**

# Functional Implementation

How would we define head, tail, and map of an ’a stream?

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

```
let rec map (f:'a->'b) (s:'a stream) : 'b stream =
  fun () -> Cons(f (head s), map f (tail s))
```

Importantly, map must return a function, which delays evaluating the recursive call to map.

# Functional Implementation

How would we define head, tail, and map of an 'a stream?

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

```
let rec map (f:'a->'b) (s:'a stream) : 'b stream =
  fun () -> Cons(f (head s), map f (tail s))
```

```
let rec ones = fun () -> Cons(1,ones)

let inc x = x + 1

let twos = map inc ones
```

# Functional Implementation

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

```
let rec map (f:'a->'b) (s:'a stream) : 'b stream =
   fun () -> Cons(f (head s), map f (tail s))
```

```
let rec ones = fun () -> Cons(1,ones)
let twos = map (fun x -> x+1) ones
```

head twos
--> head (map inc ones)
--> head (fun () -> Cons (inc (head ones), map inc (tail ones)))
--> match (fun () -> ...) () with Cons (h, _) -> h
--> match Cons (inc (head ones), map inc (tail ones)) with Cons (h, _) -> h
--> match Cons (inc (head ones), fun () -> ...) with Cons (h, _) -> h
--> ... --> 2

# Functional Implementation

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

```
let rec zip f s1 s2 =
  fun () ->
   Cons(f (head s1) (head s2),
        zip f (tail s1) (tail s2))
```

# Functional Implementation

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

```
let rec zip f s1 s2 =
  fun () ->
   Cons(f (head s1) (head s2),
        zip f (tail s1) (tail s2))
```

```
let threes = zip (+) ones twos
```

# Functional Implementation

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

```
let rec zip f s1 s2 =
  fun () ->
   Cons(f (head s1) (head s2),
        zip f (tail s1) (tail s2))
```

```
let threes = zip (+) ones twos
```

```
let rec fibs =
  fun () ->
   Cons(0, fun () ->
        Cons (1,
              zip (+) fibs (tail fibs)))
```

# Unfortunately

This is not very efficient:

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

Every time we want to look at a stream (e.g., to get the head or tail), we have to re-run the function.

# Unfortunately

This is not very efficient:

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = unit -> 'a str
```

Every time we want to look at a stream (e.g., to get the head or tail), we have to re-run the function.

```
let x = head s
let y = head s
```

rerun the *entire* underlying function as opposed to fetching the first element of a list

```
let head(s:'a stream):'a =
 match s() with
 | Cons(h,_) -> h
```

# Unfortunately

This is really, really inefficient:

```
let rec fibs =
  fun () ->
   Cons(0, fun () ->
           Cons (1,
                 zip (+) fibs (tail fibs)))
```

So when you ask for the 10th fib and then the 11th fib, we are re-calculating the fibs starting from 0...

It takes exponential time, the same way this function does:

let rec fib n = if n<2 then n else fib(n-1)+fib(n-2)

If we could *cache* or *memoize* the result of previous fibs...

# LAZY EVALUATION

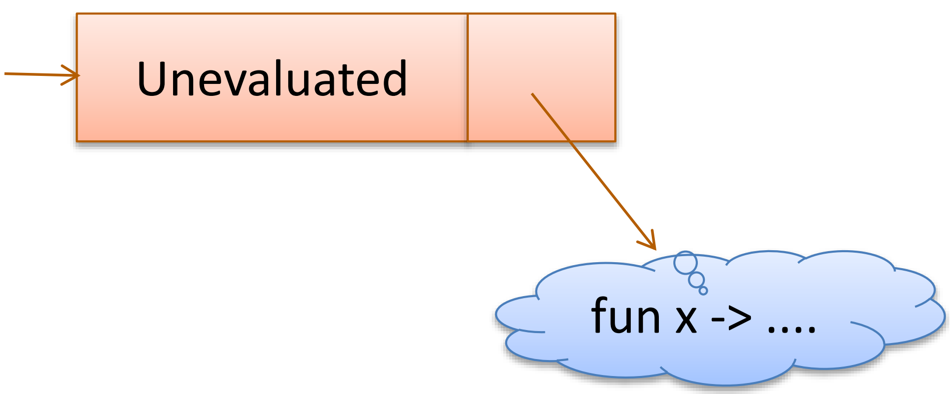
# Lazy Data

We can take advantage of mutation to memoize:

```
type ’a thunk =
  Unevaluated of (unit -> ’a) | Evaluated of ’a
```

```
type ’a lazy_t = ’a thunk ref
```

initially:




after evaluating once:

# Lazy Data

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```
type 'a lazy_t = 'a thunk ref
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```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = ('a str) lazy_t
```

# Lazy Data

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type 'a thunk =
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type 'a str = Cons of 'a * ('a stream)
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```

```
let rec head(s:'a stream):'a =
```

# Lazy Data

```
type 'a thunk =
  Unevaluated of (unit -> 'a) | Evaluated of 'a
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```
type 'a lazy_t = 'a thunk ref
```

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = ('a str) lazy_t
```

```
let rec head(s:'a stream):'a =
  match !s with
  | Evaluated (Cons(h,_)) ->
  | Unevaluated f ->
```

# Lazy Data

```
type 'a thunk =
  Unevaluated of (unit -> 'a) | Evaluated of 'a
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# Lazy Data

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type 'a thunk =
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```
let rec head(s:'a stream):'a =
  match !s with
  | Evaluated (Cons(h,_)) -> h
  | Unevaluated f ->
     let x = f() in (s := Evaluated x; head s)
```

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type 'a thunk =
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```
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```

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = ('a str) lazy_t
```

```
let rec tail(s:'a stream) : 'a stream =
  match !s with
  | Evaluated (Cons(_,t)) -> t
  | Unevaluated f ->
     (let x = f () in s := Evaluated x; tail s)
```

# Lazy Data

```
type 'a thunk =
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```
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```

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```

# Lazy Data

```
type 'a thunk =
  Unevaluated of (unit -> 'a)    Evaluated of 'a
```

```
type 'a lazy_t = '    nk
```

```
type 'a        Con
and 'a
```

```
  |
                                                    il s)
```

```
                                'a

  | Ev        d  (   s(h,_))  -> h
  | Un   luated f ->
      let x = f() in (s := Evaluated x; head s)
```

Common pattern!

Dereference & check if evaluated:

- If so, take the value.
- If not, evaluate it & take the value

# Memoizing Streams

```
type 'a thunk =
  Unevaluated of (unit -> 'a) | Evaluated of 'a
type 'a lazy_t = ('a thunk) ref

type 'a str = Cons of 'a * ('a stream)
and 'a stream = ('a str) lazy_t

let rec force(t:'a lazy_t):'a =
  match !t with
  | Evaluated v -> v
  | Unevaluated f ->
      let v = f() in
      (t:= Evaluated v ; v)

let head(s:'a stream) : 'a =
  match force s with
  | Cons(h,_) -> h

let tail(s:'a stream) : 'a stream =
  match force s with
  | Cons(_,t) -> t
```

# Memoizing Streams

```
type 'a thunk =
  Unevaluated of unit -> 'a | Evaluated of 'a

type 'a str = Cons of 'a * ('a stream)
and 'a stream = ('a str) thunk ref

let rec ones =
  ref (Unevaluated (fun () -> Cons(1,ones)))
```

# Memoizing Streams

```
type 'a thunk =
  Unevaluated of unit -> 'a | Evaluated of 'a

type 'a str = Cons of 'a * ('a stream)
and 'a stream = ('a str) thunk ref

let lazy f = ref (Unevaluated f)

let rec ones =
  lazy (fun () -> Cons(1,ones))
```

# What’s the interface?

```
type 'a lazy_t

val  lazy : (unit -> 'a) -> 'a lazy_t

val  force :  'a lazy_t -> 'a
```

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = ('a str) lazy_t

let rec ones =
  lazy(fun () -> Cons(1,ones))
```

# What’s the interface?

```
type 'a lazy_t

val  lazy : (unit -> 'a) -> 'a lazy_t

val  force :  'a lazy_t -> 'a
```

```
type 'a str = Cons of 'a * ('a stream)
and 'a stream = ('a str) lazy_t

let rec zip f s1 s2 = lazy (fun () ->
  match force s1, force s2 with
    Cons (x1,r1), Cons (x2,r2) -> Cons (f x1 x2,
                                        zip f r1 r2)
)
```

# OCaml's Builtin Lazy Constructor

If you use Ocaml's built-in Lazy.t, then you can write:

```
let rec ones = lazy (Cons(1,ones))
```

and this takes care of wrapping a "ref (Unevaluated (fun () -> ...))" around the whole thing. It has the effect of suspending the computation until you use Lazy.force

So for example:

```
let rec fibs =
  lazy (Cons(0,
    lazy (Cons(1,
      zip (+) fibs (tail fibs)))))
```

# The whole example at once

```
type 'a str = Cons of 'a * 'a stream
and 'a stream = ('a str) Lazy.t;;

let rec zip f (s1: 'a stream) (s2: 'a stream) : 'a stream =
 lazy (match Lazy.force s1, Lazy.force s2 with
        Cons (x1,r1), Cons (x2,r2) ->
                 Cons (f x1 x2, zip f r1 r2))

let tail (s: 'a stream) : 'a stream =
 match Lazy.force s with Cons (x,r) -> r

let rec fibs : int stream =
  lazy (Cons(0, lazy (Cons (1, zip (+) fibs (tail fibs)))));;

let rec printn n s =
 if n>0 then
   match Lazy.force s with
     Cons (x,r) -> (printf "%d\n" x; printn (n-1) r)

let _ = printn 10 fibs
```

# EVALUATION ORDER:
# CALL-BY-VALUE VS
# CALL-BY-NAME VS
# LAZY (a.k.a. CALL-BY-NEED)

# OCaml is Call-by-value

let x = e1 in e2

Evaluation strategy:

- evaluate e1 until you get a value
- bind that value to x
- evaluate e2 until you get a value

evaluate 2 + 3 first

Example

```
    let x = 2 + 3 in x – 7
--> let x = 5 in x – 7
--> 5 – 7
--> -2
```

# OCaml is Call-by-value

let x = e1 in e2

Evaluation strategy:

- evaluate e1 until you get a value
- bind that value to x
- evaluate e2 until you get a value

e1 e2

Evaluation strategy:

- evaluate e1 until you get a value (fun x -> e)
- evaluate e2 until you get a value (v)
- substitute v for x in e to get e’
- continue evaluating e’ until you get a value

# OCaml is Call-by-value

```
let x = e1 in e2
```

Evaluation strategy:

- evaluate e1 until you get a value
- bind that value to x
- evaluate e2 until you get a value

```
e1 e2
```

Evaluation strategy:

- evaluate e1 until you get a value (fun x -> e)
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*Is this the only way we could evaluate these expressions?*
*Is this the most efficient way we could evaluate these expressions?*

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Evaluation strategy:

- evaluate e1 until you get a value (fun x -> e)
- evaluate e2 until you get a value (v)
- substitute v for x in e to get e’
- continue evaluating e’ until you get a value

*Is this the only way we could evaluate these expressions?* ***No!***
*Is this the most efficient way we could evaluate these expressions?* ***No!***

# Call-by-Name

```
let x = e1 in e2
```

Evaluation strategy:

- bind that expression e1 to x
- continue to evaluate e2

Example

```
    let x = 2 + 3 in x – 7
--> (2 + 3) – 7
--> 5 – 7
--> -2
```

# Call-by-Name

```
let x = e1 in e2
```

Evaluation strategy:

- bind that expression e1 to x
- continue to evaluate e2

Call-by-name
can avoid
work sometimes:

```
    let x = work () in 7
--> 7
```

# Call-by-Name

let x = e1 in e2

Evaluation strategy:

- bind that expression e1 to x
- continue to evaluate e2

Call-by-name
can avoid *A LOT* of
work sometimes:

```
    let x = loop_forever () in 7
--> 7
```

# Call-by-Name

let x = e1 in e2

Evaluation strategy:

- bind that expression e1 to x
- continue to evaluate e2

But sometimes it does *more* work than necessary

```
    let x = work () in x + x
--> (work ()) + (work ())
```

# Call-by-Name (CBN) vs Call-by-Value (CBV)

In general:

CBV can be asymptotically faster than CBN (by exponential factor at least!)

CBN can be asymptotically faster than CBV (by exponential factor at least!)

However:

CBV can diverge (infinite-loop) where CBN terminates but not vice versa!
If CBN diverges, then ANY strategy diverges

Therefore:

CBN is the "most general" strategy, in the sense that it terminates as often as possible. Though it definitely isn't necessarily fastest!

by the way, guess who figured all this out:
Alonzo Church and his graduate students, Princeton University, 1930s

# Call-by-Name vs Lazy

```
let x = e1 in e2
```

Lazy evaluation is like call-by-name but it avoids repeatedly executing e1 by using *memoization* – it computes an answer once and then remembers the result if x is ever needed a 2nd or 3rd time

The operational semantics notation is less compact when it comes to describing lazy computations because we have to keep track of the imperative state used for memoization. So I won't try here.

```
    let x = work () in x + x
--> ...
--> ...
```

# Call-by-Name vs Lazy vs Call-by-Value

In general:

LAZY can be asymptotically faster than CBN.

– thanks to memoization – no repeated calls

CBN is never asymptotically faster than LAZY.

CBN terminates if-and-only-iff LAZY terminates.

(Thus) LAZY is *also* a most-general strategy.

In practice:

- Data structures used to memoize computations take up space
  - thunks hang on to data structures, making it tough to reason about
- Much optimization needed for CBN to approach CBV performance
- But laziness (“deferred, call-by-need computation”) can be useful
  - we can program with selective laziness in call-by-value languages

# Historical note

The term “thunk” was introduced in 1960 to describe a way of implementing call-by-name function-parameters in Algol 60,

by P. Z. Ingerman of U. Penn.

An early form of “closures” (for environments of thunks) was invented by:

Ned Irons ’58 of Princeton University and Wally Feurzig of U. Chicago.

Ned Irons was one of the most brilliant programmers of his time.

Wally Feurzig was a pioneer in A.I. and in C.S. education, a collaborator of Turing award winners Marvin Minsky (PhD Princeton 1951) and John McCarthy (PhD Princeton 1951).


1. P.Z. Ingerman, "Thunks: a way of compiling procedure statements with some comments on procedure declarations". *Communications of the ACM* **4** (1): 55–58, 1961.
2. E. T. Irons and W. Feurzig, "Comments on the Implementation of Recursive Procedures and Blocks in ALGOL". *Communications of the ACM* **4** (1): 65–69, 1961.

# Summary

By default, OCaml (and Java, C, etc) is an eager language

- but you can use thunks or “lazy” to suspend computations
- use “force” to run the computation when needed

By default, Haskell is a lazy language

- the implementers (eg: Simon Peyton Jones) would probably make it eager by default if they had a do-over
- working with infinite data is generally more pleasant
- but difficult to reason about space and time

Lazy evaluation makes it possible to build *infinite data structures*.

- can be modelled using functions
- but adding refs allows memoization