COS 226 **Algorithms and Data Structures** **Spring 2021**

Midterm Solutions

1. **Initialization.** Don't forget to do this.

2. **Memory.**

   (a) 32

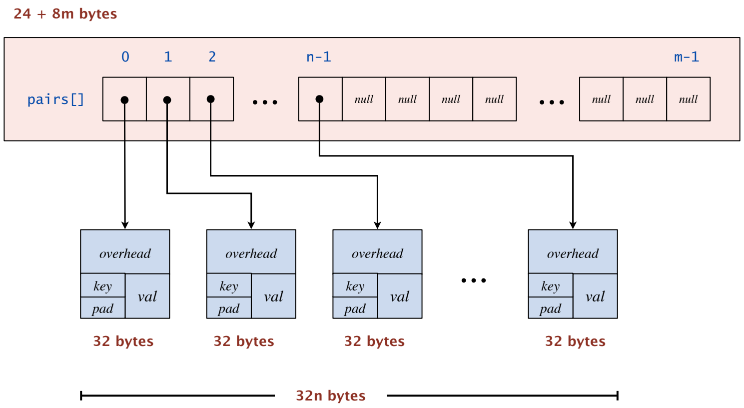
   Each `Pair` *object uses* 32 *bytes of memory:*

   - 16 bytes of object overhead
   - 4 bytes for the integer `key`
   - 8 bytes for the double `value`
   - 4 bytes of padding (to make memory usage a multiple of 8)

   (b) $\sim 40n$

   A `SortedArray` *object uses* $\sim 8m + 32n$ *bytes of memory when the array* `pairs[]` *is of length* m *and has* n *non-null entries.*

   - 16 bytes of object overhead
   - 4 bytes for the integer `n`
   - 4 bytes of padding
   - 8 bytes for the reference to `pairs[]`
   - $24 + 8m$ bytes for the array of references of length m.
   - $32n$ bytes for the n `Pair` objects.



   When the array is full, $m = n$*, which is* $\sim 40n$.

   (c) $\sim 64n$

   When the array is one-quarter full, $m = 4n$*, which is* $\sim 64n$.

3. **Five sorting algorithms.**

   (3.1) *selection sort after 12 iterations*

   (3.2) *mergesort just before the last call to* `merge()`

   (3.3) *insertion sort after 16 iterations*

   (3.4) *quicksort after first partitioning step*

   (3.5) *heapsort immediately after the heap construction phase*

4. **Analysis of algorithms.**

(4.1) $\sim \frac{1}{2}n^2$

Selection sort makes $\sim \frac{1}{2}n^2$ *compares to sort any array of length* n*.*

(4.2) $\sim \frac{1}{8}n^2$

Let's consider the even and odd iterations of insertion sort separately.

- *In the odd iterations (when we are inserting the integers* $1, 2, \ldots, n$*), there are no exchanges because each integer is larger than every element to its left.*
- *In the even iterations (when we are inserting the 0s), the 0 must be exchanged with all of the positive integers to its left.*

So, the total number of exchanges is $0 + 1 + 2 + \ldots + (n/2 - 1) \ \sim \frac{1}{8}n^2$*.*
The number of compares in insertion sort is always within an additive factor of n *of the number of exchanges.*

(4.3) $\sim \frac{3}{4}n \log_2 n$

In each merge, the left subarray contains $n/4$ *0s followed by* $n/4$ *smaller integers; and the right subarray contains* $n/4$ *0s followed by* $n/4$ *larger integers. Here is an example when* $n = 4$*:*

$$0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \;\Big|\; 0 \quad 0 \quad 0 \quad 0 \quad 5 \quad 6 \quad 7 \quad 8$$

Merging two subarrays of this form involves $\sim \frac{3}{4}n$ *compares because the left subarray is exhausted before taking any of the integers from the right subarray. This is true at every level in the recursion.*

(4.4) $\Theta(n)$, $O(n)$, $O(n \log n)$, $O(n^2)$

$f(n) = n + \frac{1}{2}n + \frac{1}{4}n + \frac{1}{8}n + \ldots + 1 \ = 2n - 1.$

Big O and big Theta notations discard both lower-order terms and the leading coefficient. The main difference is that big O notation includes functions that grow more slowly. So, $O(n \log n)$ *includes not only functions like* $2n \log_2 n$ *and* $\frac{1}{2}n \log_2 n$*, but also* $2n - 1$ *and* $\frac{1}{8}\sqrt{n}$*.*

5. Predecessor search in a BST.

C D C E F D C

This is identical to the `floor()` *function from lecture, except for when the search key is equal to the key in the node, in which case you should find the predecessor in the left subtree.*

```
private Key pred(Node x, Key k, Key champ) {
    if (x == null) return champ;
    int cmp = k.compareTo(x.key);
    if      (cmp < 0) return pred(x.left,  k, champ);
    else if (cmp > 0) return pred(x.right, k, x.key);
    else              return pred(x.left,  k, champ);
}
```

6. Mystery key.

(6.1) 65, 70

The constraints of the binary heap imply that $60 \leq x \leq 80$.

(6.2) 0, 6, 7

When it is time for E to be inserted, the linear-probing hash table will have the following structure:

0	1	2	3	4	5	6	7
		B		A		C	D

(6.3) (55, 65), (75, 55)

The constraints of the k-d tree imply that $50 \leq x \leq 80$ *and that* $40 \leq y \leq 70$.

7. Why did Java do that?

(7.1) X O X O O X

Like mergesort, in the worst case, Timsort makes $\sim n \log_2 n$ *compares and uses* $\Theta(n)$ *extra space. Timsort is optimized for input arrays that have a small number of runs (either in increasing or decreasing order); it makes* $\Theta(n)$ *compares in such cases.*

(7.2) X X X O O X

A doubly linked list supports adding/removing from the front or back in $\Theta(1)$ *time, as in a* `Deque`*. Accessing the element at index* $n/2$ *takes* $\Theta(n)$ *time. Each iterator uses* $\Theta(1)$ *extra memory—it only needs to maintain a reference to the current node in the iteration.*

8. **Triple sum.**

The main idea is to put the integers in `c[]` into a hash table and then iterate over all pairs of elements `a[i]` and `b[j]` and check whether `-(a[i] + b[j])` is in the hash table. This solution takes $O(n^2)$ time on typical inputs and uses $\Theta(n)$ extra space.

Here's the corresponding Java code.

```
public boolean hasTripleSum(int n, long[] a, long[] b, long[] c) {

    // add elements of c[] to hash table
    // key = integer, value = array index (but not used here)
    HashMap<Long, Integer> st = new HashMap<>();
    for (int k = 0; k < n; k++)
        st.put(c[k], k);

    // for each a[i] and b[j], check whether it sums to 0 with an integer from c[]
    for (int i = 0; i < n; i++)
        for (int j = 0; j < n; j++)
            if (st.containsKey(-(a[i] + b[j])))
                return true;

    return false;
}
```

Here's an alternative solution that uses sorting and a carefully orchestrated search. This solution takes $\Theta(n^2)$ *time in the worst case and uses only* $\Theta(1)$ *extra memory.*

```
public static boolean hasTripleSum(int n, long[] a, long[] b, long[] c) {
    Heap.sort(a);  // assume overloaded method for sorting long[]
    Heap.sort(b);

    // for each c[k], check whether it sums to 0 with an integer from a[] and b[]
    for (int k = 0; k < n; k++) {
        int i = 0, j = n - 1;
        while (i < n && j >= 0) {
            if      (a[i] + b[j] + c[k] > 0) j--;
            else if (a[i] + b[j] + c[k] < 0) i++;
            else return true;
        }
    }
    return false;
}
```

9. **Multiset data type.**

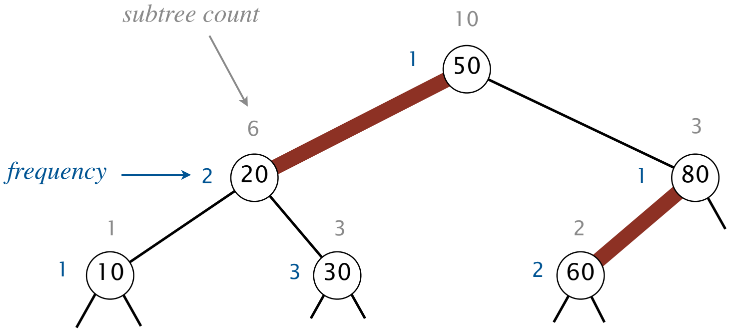
Half-credit solution. *Create a* `RedBlackBST<Long, Integer>` *where the key is the integer in the multiset and the corresponding value is the number of times that integer appears.*

```
public class MultisetHalfCredit {
    private RedBlackBST<Long, Integer> st = new RedBlackBST<>();

    // add k to the multiset
    public void add(long k) {
        if (st.contains(k)) st.put(k, st.get(k) + 1);
        else st.put(k, 1);
    }

    // number of integers in the multiset equal to k
    public int count(String item) {
        if (st.contains(k)) return st.get(k);
        else return 0;
    }

    // number of integers in the multiset strictly less than k
    public int rank(int k) {
        int sum = 0;
        for (long i : st.keys())
            if (i < k) sum += st.get(i);
            else break;
        return sum;
    }
}
```

The `add()` *and* `count()` *methods take* $\Theta(\log n)$ *time in the worst case. However, the* `rank()` *method takes* $\Theta(n)$ *time in the worst case.*

Full-credit solution. *The main idea to achieve a* $\Theta(\log n)$ *performance guarantee for* `rank()` *is to re-implement a* red–black BST*, keeping the frequency counts in the nodes* and *modifying the subtree counts so that they account for duplicate integers.*

For example, here a red–black BST corresponding to a multiset with the 10 integers 10, 20, 20, 30, 30, 30, 50, 60, 60, 80.



The specific modifications to `RedBlackBST` are as follows:

- *Value field.* Replace the `value` field with a frequency count field.
- *Add method.* Same as `put()` in a red–black BST except that if the key to be added is not in the BST, it sets the frequency in the node to 1; if the key is already in the BST, it increments the frequency in the corresponding node by 1.
- *Count method.* Same as `get()` in a BST except that it returns the frequency of the key in the BST, and 0 otherwise.
- *Rank method.* Same as `rank()` in a BST except to account for duplicate keys. Specifically, if the search key is greater than the key in the node, it should return

  ```
  size(x.left) + rank(key, x.right) + x.frequency;
  ```

  instead of

  ```
  size(x.left) + rank(key, x.right) + 1;
  ```

- *Maintaining subtree counts.* Same as in `add()`, `rotateLeft()`, and `rotateRight()` except to account for duplicate keys. Specifically, when updating the subtree counts, use

  ```
  x.size = size(x.left) + size(x.right) + x.frequency;
  ```

  instead of

  ```
  x.size = size(x.left) + size(x.right) + 1;
  ```

For reference, here's the relevant Java code. All instance methods take $\Theta(\log n)$ *time in the worst case.*

```
public class Multiset {
    private Node root;

    private class Node {
        private long key;          // key
        private int frequency;     // number of occurrences of key
        private Node left, right;  // links to left and right subtrees
        private boolean color;     // color of parent link
        private int size;          // subtree count
    }

    private Node add(Node x, long key) {
        if (x == null) return new Node(key, RED);

        if      (key < x.key) x.left  = add(x.left,  key);
        else if (key > x.key) x.right = add(x.right, key);
        else                  x.frequency++;

        if (isRed(x.right) && !isRed(x.left))      x = rotateLeft(x);
        if (isRed(x.left)  &&  isRed(x.left.left)) x = rotateRight(x);
        if (isRed(x.left)  &&  isRed(x.right))     flipColors(x);

        x.size = size(x.left) + size(x.right) + x.frequency;

        return x;
    }

    public int count(long key) {
        Node x = root;
        while (x != null) {
            if      (key < x.key) x = x.left;
            else if (key > x.key) x = x.right;
            else return x.frequency;
        }
        return 0;
    }

    public int rank(long key) {
        return rank(key, root);
    }

    // number of keys less than key in the subtree rooted at x
    private int rank(long key, Node x) {
        if (x == null) return 0;
        if      (key < x.key) return rank(key, x.left);
        else if (key > x.key) return size(x.left) + rank(key, x.right) + x.frequency;
        else                  return size(x.left);
    }

}
```

Alternative full-credit solution. Modify the `RedBlackBST` to allow duplicate keys. The key modifications are as follows:

- *Value field.* No need for the `value` field.
- *Add method.* Same as `put()` in a red–black BST except that if the key to be added is equal to the key in the node, insert it in the left subtree.
- *Rank method.* Same as `rank()` in a BST except that if the search key is equal to the key in the node, return the rank in the left subtree.
- *Count method.* Not easy to do efficiently directly because the equal keys can be scattered throughout the BST. But, it's easy to implement `count()` efficiently with two calls to `rank()`.

```
public int count(int k) {
    return rank(k + 1) - rank(k);
}
```

This solution uses $\Theta(n)$ *space, where* n *is the number of integers in the multiset. The other solutions uses* $\Theta(m)$ *space, where* m *is the number of* **distinct** *integers in the multiset.*