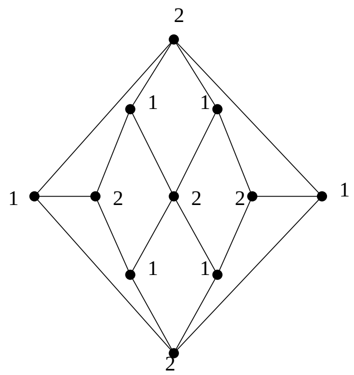
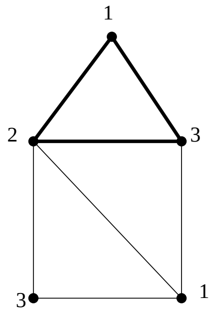
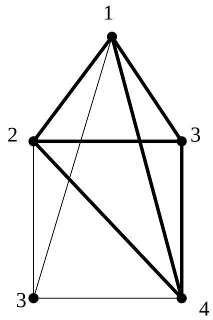
COS 341 Homework 9 Solutions

Section 13.6

Problem 5. The chromatic numbers are 2, 3, and 4 respectively. To prove this, you need to provide both an upper bound (an example coloring) and a lower bound (a reason why a clever person cannot do better). The leftmost graph obviously requires at least 2 colors. The middle graph contains a complete 3 node graph as a subgraph, and so requires at least 3 colors. The rightmost graph contains a complete 4 node graph, and so requires at least 4 colors. The upper bounds are given below:







Problem 13. We prove this by induction using the discussion on pages 510–511. For the base case, when $n = 3$ the graph is a triangle, which has chromatic polynomial $p_G(k) = k(k-1)(k-2) = (k-1)^3 - (k-1)$ (because each vertex must get a different color). Assume that the given expression is true for $n-1$, we'll show it's true for n. Let $G = C_n$ be a cycle of order n. Choose any edge and apply (13.1). We have that

$$p_G(k) = p_{G_1}(k) - p_{G_2}(k)$$

where G_1 is a tree of order n and G_2 is a cycle of order $n-1$. By the induction hypothesis we have

$$p_G k = p_{G_1}(k) - (k-1)^{n-1} - (-1)^{n-1}(k-1)$$

By Theorem 13.1.7 we have:

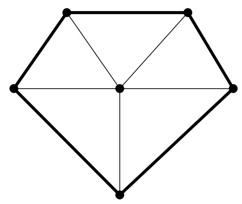
$$p_G(k) = k(k-1)^{n-1} - (k-1)^{n-1} - (-1)^{n-1}(k-1)$$

Rearranging,

$$\begin{aligned} &= (k-1)(k-1)^{n-1} + (-1)^n(k-1) \\ &= (k-1)^n + (-1)^n(k-1) \end{aligned}$$

which is the desired result.

Problem 18. Below is one possibility:

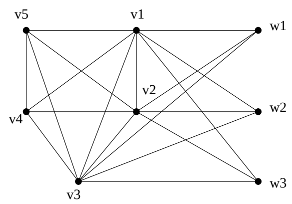


To see that this requires 4 colors note that whatever color is used in the center vertex cannot be used in the shaded path. This shaded path is a cycle of odd length and thus requires 3 colors. Hence, the total graph requires 4 colors.

Problem 30. A tree T has no cycles and thus is bipartite (since it clearly has no odd cycles). The odd levels of the tree form one independent set and the even levels form another. One of those two sets must have at least $\lceil n/2 \rceil$ nodes in it (since if both sets had $< \lceil n/2 \rceil$ there would be $< n$ nodes in the tree). Thus, $\alpha(T) \geq \lceil n/2 \rceil$.

Problem 42. Consider a subgraph H of a bipartite graph G. If H is null (contains no edges) then $\chi(H) = 1 = \omega(H)$. If H is not null, then it is bipartite (because all non-null subgraphs of a bipartite graph are themselves bipartite). In this case $\chi(H) = 2$ (color the left half red and the right half blue). Also, $\omega(H) = 2$ because there are only edges going from the left half to the right half. Since $\chi(H) = \omega(H)$ for all subgraphs H of G, the bipartite graph G is by definition perfect.

Special Problem 1. The graph is as shown:



Part (a) $\omega(G) = 5$ because the v_i form a clique of size 5 and no other vertices have degree ≥ 5.

Part (b) $\alpha(G) = 4$; choose $\{w_1, w_2, w_3, v_4\}$, for example. No bigger independent set exists because you can only choose at most one vertex v_i and at most one side of the complete bipartite subgraph $K_{3,3}$ formed by $v_1, v_2, v_3, w_1, w_2, w_3$.

Part (c) $\chi(G) = 5$. You know $\chi(G) \geq 5$ because $\omega(G) = 5$ (part (a)). An 5 coloring can be obtained by coloring v_1, v_2, v_3, v_4, v_5 with colors a, b, c, d, e respectively, and coloring w_1, w_2, w_3 with color e.

Part (d) G is not planar because it contains K_5 as a subgraph.

Part (e) G is not Eulerian because several vertices have odd degree (see Theorem 11.2.2).

Part (f) G does not contian a Hamiltonian cycle. To see this look at the bipartite subgraph formed by $v_1, v_2, v_3, w_1, w_2, w_3$. Any Hamiliton cycle in G must include a Hamiliton chain through this subgraph. Further, that chain must begin and end at one of the vertices v_1, v_2, v_3. By Theorem 11.4.2, this is impossible.