Homework 8 Solutions

December 12, 2001

Section 9.5 #20, 26; Section 11.8 #9, 12, 20, 40

9.5.20 Suppose that A is not paired with a. But then A marries some man b and a marries some woman C, yet A and a prefer the other over the person they married. By definition this is an unstable marriage, a contradiction.

9.5.26 We apply the deferred acceptance algorithm to the following preference matrix:

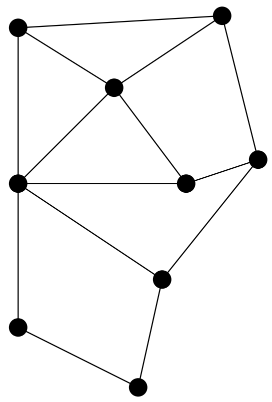
$$\begin{array}{cc} & \begin{array}{cccc} \text{a} & \text{b} & \text{c} & \text{d} \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 1,3 & 2,2 & 3,1 & 4,3 \\ 1,4 & 2,3 & 3,2 & 4,4 \\ 3,1 & 1,4 & 2,3 & 4,2 \\ 2,2 & 3,1 & 1,4 & 4\ 1 \end{bmatrix} \end{array}$$

Following the algorithm we have:

1. A asks a, B asks a, C asks b, D asks c, a rejects B
2. B asks b, b rejects C
3. C asks c, c rejects D
4. D asks a, a rejects A
5. A asks b, b rejects B
6. B asks c, c rejects C
7. C asks a, a rejects D
8. D asks b, b rejects A
9. A asks c, c rejects B
10. B asks d.

We obtain the matching: $A \leftrightarrow c$, $C \leftrightarrow a$, $B \leftrightarrow d$, $D \leftrightarrow d$.

11.8.9 One example:



11.8.12 The left and middle graphs are isomorphic, shown by specifying a mapping from one graph to another. The right graph is not isomorphic to the others, as it includes cycles of length 4, whereas the first two have smallest cycles of length 5.

11.8.20 If our graph G of order n is not connected, then there are at least 2 disconnected subgraphs. Say that the subgraph G_1 has order k, and G_2 has order $n-k$. The maximum number of edges possible in G is $k(k-1)/2+(n-k)(n-k-1)/2 = \frac{1}{2}(n^2-n(2k+1)+2k^2$. Since $1 \leq k \leq n-1$, this is at most $(n-1)(n-2)/2$ (when $k=1$ or $k=n-1$), which contradicts the requirement that G has $(n-1)(n-2)/2+1$ edges.

One example of a disconnected graph of order n with $(n-1)(n-2)/2$ edges is a graph with a complete subgraph of order $n-1$, plus one unconnected vertex.

Several people counted edges between two unconnected components, removed edges from a complete graph of order n, or applied induction. Regardless of the method, it was not sufficient to show that a graph with K_{n-1} plus one vertex had to be connected, since this does not represent all possible graphs with $(n-1)(n-2)/2+1$ edges. (Likewise for the next problem.)

11.8.40 We show that G satisfies the Ore property. Then, we can apply Theorem 11.3.2. Consider the subgraph G' of G that does not include edges u and v. The maximum number of edges in G' is $(n-2)(n-3)/2$, if it is complete. If u and v are not adjacent, then $deg(u)+deg(v) \geq (n-1)(n-2)/2+2-(n-2)(n-3)/2 = n$, satisfying the Ore property.

A example of a graph of order n with $(n-1)(n-2)/2+1$ edges, without a Hamiltonian cycle, is one with a complete subgraph of order $n-1$, connected by one edge to a remaining vertex.