COS 341, October 24, 2001
Handout Number 8

Solving Recurrence with Generating Functions II

The next problem we tackle is the recurrence $a_0 = 0$, $a_1 = 1$, and $a_n = \sum_{1 \leq i \leq n-1} a_i a_{n-i}$ for $n \geq 2$. The quantity a_n is the number of ways to parenthesize an expression $y_1 + y_2 + \cdots + y_n$.

Let $A(x) = \sum_{n \geq 0} a_n x^n = \sum_{n \geq 1} a_n x^n$. The recurrence relation gives

$$\begin{aligned}
\sum_{n \geq 2} a_n x^n &= \sum_{n \geq 2} \sum_{1 \leq i \leq n-1} a_i x^i a_{n-i} x^{n-i} \\
&= (\sum_{i \geq 1} a_i x^i)(\sum_{j \geq 1} a_j x^j) \\
&= (A(x))^2.
\end{aligned}$$

This means $A(x) - x = (A(x))^2$, and hence $A(x)^2 - A(x) + x = 0$. Solving the quadratic equation for $A(x)$, we obtain two possible solutions: $A(x) = (1 + \sqrt{1-4x})/2$ and $A(x) = (1 - \sqrt{1-4x})/2$. The former solution can be discarded, since it would give $a_0 = A(0) = 1$, which contradicts our assumption $a_0 = 0$. Thus,

$$\begin{aligned}
A(x) &= \frac{1 - \sqrt{1-4x}}{2} \\
&= \frac{1}{2} - \frac{1}{2}(1-4x)^{1/2} \\
&= \frac{1}{2} - \frac{1}{2} \sum_{n \geq 0} \binom{1/2}{n} (-4x)^n.
\end{aligned}$$

We infer from it $a_0 = 0$, and for $n \geq 1$,

$$a_n = -\frac{1}{2} \binom{1/2}{n} (-4)^n.$$

Note that, for $n \geq 2$

$$\begin{aligned}
\binom{1/2}{n} &= \frac{1}{n!} \frac{1}{2} \left(\frac{1}{2} - 1\right) \left(\frac{1}{2} - 3\right) \cdots \left(\frac{1}{2} - (n-1)\right) \\
&= \frac{1}{n!} \frac{1}{2} \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \cdots \left(-\frac{2n-3}{2}\right) \\
&= \frac{1}{2^n} \frac{1}{n!} (-1)^{n-1} 1 \cdot 3 \cdots (2n-3) \\
&= \frac{1}{2^n} \frac{1}{n!} (-1)^{n-1} \frac{1 \cdot 2 \cdot 3 \cdots (2n-2)}{2 \cdot 4 \cdots (2n-2)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^n}\frac{1}{n!}(-1)^{n-1}\frac{(2n-2)!}{2^{n-1}(n-1)!} \\
&= (-1)^{n-1}\frac{2}{4^n}\frac{1}{n}\binom{2n-2}{n-1}.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
a_n &= -\frac{1}{2}\binom{1/2}{n}(-4)^n \\
&= -\frac{1}{2}(-4)^n \cdot (-1)^{n-1}\frac{2}{4^n}\frac{1}{n}\binom{2n-2}{n-1} \\
&= \frac{1}{n}\binom{2n-2}{n-1},
\end{aligned}
$$

for $n \geq 2$. The above formula also holds for $n = 1$ since both sides are equal to 1. (Recall $\binom{0}{0} = \frac{0!}{0!0!} = 1$.) The numbers a_n are often called the *Catalan numbers*.

System of Recurrences

We now consider a problem that involves solving more than one recurrence relations. Let a_n be the number of ways to tile with dominos a $3 \times n$ rectangle, and b_n be the number of ways to tile with dominos a $3 \times n$ *quasi-rectangle* (ie, rectangle with one corner square missing). From discussions in class, we have $a_1 = 0, a_2 = 3, b_1 = 1, b_2 = 0$, and for all $n \geq 3$,

$$
\begin{aligned}
a_n &= a_{n-2} + 2b_{n-1}, \\
b_n &= a_{n-1} + b_{n-2}.
\end{aligned} \tag{1}
$$

Let $a_0 = b_0 = 0$. Consider the generating function $A(x) = \sum_{n\geq 0} a_n x^n$, and $B(x) = \sum_{n\geq 0} b_n x^n$. Then from (1) we obtain

$$
\sum_{n\geq 3} a_n x^n = \sum_{n\geq 3} a_{n-2}x^n + 2\sum_{n\geq 3} b_{n-1}x^n.
$$

This leads to

$$
A(x) - a_0 - a_1 x - a_2 x^2 = x^2(A(x) - a_0) + 2x(B(x) - b_0 - b_1 x). \tag{2}
$$

Similarly, we obtain

$$
B(x) - b_0 - b_1 x - b_2 x^2 = x(A(x) - a_0 - a_1 x) + x^2(B(x) - b_0). \tag{3}
$$

Substituting the values of $a_0, a_1, a_2, b_0, b_1, b_2$ into (2) and (3), we obtain after rearranging terms,

$$
\begin{aligned}
(1-x^2)A(x) - 2xB(x) &= x^2 && (4)\\
xA(x) - (1-x^2)B(x) &= -x. && (5)
\end{aligned}
$$

We now solve (4) and (5) for $A(x), B(x)$. From (5) we have

$$B(x) = \frac{x}{1-x^2}(A(x)+1). \tag{6}$$

Substituting (6) into (4) we get

$$(1-x^2)A(x) - 2x\frac{x}{1-x^2}(A(x)+1) = x^2.$$

This leads immediately to

$$A(x) = \frac{x^2(3-x^2)}{(1-x^2)^2 - 2x^2} = \frac{1-x^2}{(1-x^2)^2-2x^2} - 1. \tag{7}$$

It remains to extract a_n from $A(x)$. Let $y = x^2$. We have

$$A(x) = \frac{1-y}{1-4y+y^2} - 1.$$

Using the partial fraction decomposition method as before, we obtain

$$
\begin{aligned}
A(x) &= (1-y)\frac{1}{(1-(2+\sqrt{3})y)(1-(2-\sqrt{3})y)} - 1\\
&= (1-y)\frac{1}{2\sqrt{3}}\left(\frac{2+\sqrt{3}}{(1-(2+\sqrt{3})y)} - \frac{2-\sqrt{3}}{(1-(2-\sqrt{3})y)}\right) - 1\\
&= (1-y)\frac{1}{2\sqrt{3}}\sum_{n\geq 0}((2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1})y^n - 1\\
&= (1-x^2)\frac{1}{2\sqrt{3}}\sum_{n\geq 0}((2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1})x^{2n} - 1\\
&= \frac{1}{2\sqrt{3}}\sum_{n\geq 0}((2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1})x^{2n} -\\
&\qquad \frac{1}{2\sqrt{3}}\sum_{n\geq 0}((2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1})x^{2(n+1)} - 1.
\end{aligned}
$$

Since a_m is equal to the coefficient of the x^m term in the above expression, we conclude that for all $n \geq 1$,

$$
\begin{aligned}
a_{2n-1} &= 0,\\
a_{2n} &= \frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}) - \frac{1}{2\sqrt{3}}((2+\sqrt{3})^{n} - (2-\sqrt{3})^{n}).
\end{aligned}
$$

This solves the problem.