COS 341, October 5, 2001
Handout Number 4

# Math Casino

Let $N = 10^4$. Roulettes in the Math Casino have the following rule. For each run, a random integer $n$, $1 \leq n \leq N$ is uniformly generated. You pay \$1 to play. If $\lfloor n^{1/4} \rfloor$ divides $n$, you get \$5; otherwise, you get nothing. Should you play the game? One way to make a rational decision is to calculate the probability $p$ of winning a game, and play the game if and only if $p \geq 1/5$. How can we compute $p$?

By definition $p = m/N$, where $m$ is the number of integers $1 \leq n \leq 10^4$ that satisfy "$\lfloor n^{1/4} \rfloor$ divides $n$". Let $A_j$ be the set of integers $n$ satisfying $\lfloor n^{1/4} \rfloor = j$ (i.e., $j^4 \leq n \leq (j+1)^4 - 1$ ) and "$j$ dividing $n$". Clearly,

$$m = \sum_{1 \leq j \leq 9} |A_j| + 1,$$

where the term 1 comes from the $n = 10^4$ contribution.

Let $j \geq 1$. Starting at $n = j^4$, every $j$-th integer in the range $j^4 \leq n \leq (j+1)^4 - 1$ satisfies $j$ dividing $n$. Hence,

$$\begin{aligned}
|A_j| &= \lceil \frac{(j+1)^4 - 1 - j^4 + 1}{j} \rceil \\
&= \lceil 4j^2 + 6j + 4 + \frac{1}{j} \rceil \\
&= 4j^2 + 6j + 4 + 1 \\
&= 4j^2 + 6j + 5.
\end{aligned}$$

This shows

$$\begin{aligned}
m &= 1 + \sum_{1 \leq j \leq 9} (4j^2 + 6j + 5) \\
&= 1 + 4 \sum_{1 \leq j \leq 9} j^2 + 6 \sum_{1 \leq j \leq 9} j + 9 \cdot 5 \\
&= 46 + 4 \sum_{1 \leq j \leq 9} j^2 + 6 \frac{1}{2} 9 \cdot 10 \\
&= 316 + 4 \sum_{1 \leq j \leq 9} j^2. \qquad (1)
\end{aligned}$$

It remains to evaluate $a \equiv \sum_{1 \leq j \leq 9} j^2$. Using the equality $\sum_{2 \leq k \leq s} \binom{k}{2} = \binom{s+1}{3}$, we have

$$a = \sum_{1 \leq j \leq 9} j^2$$

$$
\begin{aligned}
&= \sum_{1\le j\le 9} (2\binom{j}{2} + j) \\
&= 2\sum_{1\le j\le 9} \binom{j}{2} + \frac{1}{2} 9\cdot 10 \\
&= 2\binom{9+1}{3} + 45 \\
&= 285.
\end{aligned}
$$

From (1) we then have

$$m = 316 + 4 \cdot 285 = 1456.$$

Hence $p = m/N = 0.1456 < 1/5$, and you should not play this game. This solves the Math Casino Problem.