COS 341, September 26, 2001
Handout Number 3

A Dating Problem

Probability

A *probability space* Ω is a pair (U, p) where U is a finite set, and $p : U \to [0, 1]$ is a function such that $\sum_{u \in U} p(u) = 1$. An *event* T is a subset of U. The *probability for* the event T *to occur* is defined as $\Pr\{T\} = \sum_{u \in T} p(u)$.

The notion of probability space is used to model a random process, in which U is the set of all possible configurations, and configuration u is to be the realized configuration with probability $p(u)$. We restrict U to be a finite set for simplicity (in general U could be infinite). The notion of event models a boolean predicate that is of interest.

A Dating Problem

In the Dating Problem with parameters n, k, the probability space is $\Omega = (U, p)$, where U is the set of all permutations of $\{1, 2, \cdots, n\}$ (hence $|U| = n!$), and $p(u) = 1/|U|$ for all $u \in U$. It is easy to see that a permutation $u = (i_1, i_2, \cdots, i_n) \in U$ is in T if and only if the following conditions are satisfied:

C1: $1 \in \{i_{k+1}, i_{k+2}, \cdots, i_n\}$;
C2: Let $i_j = 1$ where $k < j \leq n$, then the minimum of $i_1, i_2, \cdots, i_{j-1}$ is among the first k of these numbers.

Let T_j denote the set of all u's satisfying C1, C2 with j being the j in C2. Then T is the disjoint union of such T_j. By the Addition Principle, we have

$$|T| = \sum_{k < j \leq n} |T_j|. \tag{1}$$

We assert that for each $k < j \leq n$,

$$|T_j| = \frac{k}{j-1}(n-1)!. \tag{2}$$

To prove (2), note that an element $(i_1, i_2, \cdots, i_n)$ of T_j can be specified by a pair (I_1, I_2). In the first step, we specify I_1, an $(n-j)$-permuation of $\{2, 3, \cdots, n\}$, to fix the values of $i_{j+1}, i_{j+2}, \cdots, i_n$. Let L be the set $\{2, 3, \cdots, n\} - \{i_{j+1}, i_{j+2}, \cdots, i_n\}$. In the second step, we specify I_2, a $(j-1)$-permutation $\{i_1, i_2, \cdots, i_{j-1}\}$ of the set L satisfying the condition C2, in order to fix $i_1, i_2, \cdots, i_{j-1}$. The number of choices c_1 in step 1 is

$P(n-1, n-j) = (n-1)!/(n-1-(n-j))! = (n-1)!/(j-1)!$. Let the number of choices in step 2 be c_2. We prove later that

$$c_2 = (j-1)!\frac{k}{j-1}. \tag{3}$$

By the Multiplication Principle, we have

$$\begin{aligned} |T_j| &= c_1 c_2 \\ &= \frac{(n-1)!}{(j-1)!}(j-1)!\frac{k}{j-1} \\ &= (n-1)!\frac{k}{j-1}. \end{aligned} \tag{4}$$

It follows from (1), (2) and (4) that

$$|T| = (n-1)!k(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n-1}).$$

Since $|U| = n!$, this means

$$\Pr\{T\} = |T|/|U| = \frac{k}{n}(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n-1}). \tag{5}$$

It remains to prove (3). How many permutations of $j-1$ elements have their minimum located in the first k positions? Let W be the set of all permuations $(i_1, i_2, \cdots, i_{j-1})$ of $\{1, 2, \cdots, j-1\}$ such that $i_s = 1$ for some $1 \le s \le k$. We need to prove $|W| = (j-1)!\frac{k}{j-1}$. Let W_s be those $(i_1, i_2, \cdots, i_{j-1})$ with $i_s = 1$. By the Addition Principle, $|W| = \sum_{1 \le s \le k} |W_s|$. Now, clearly, $|W_s| = (j-2)!$. Hence $|W| = (j-2)!k = (j-1)!\frac{k}{j-1}$. This proves (3).

This completes the analysis of the probability of success under the strategy used with parameters n, k.

For example, if $n = 8, k = 4$, the probability of winning is by (5)

$$p_{n,k} = \frac{4}{8}(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}) = 319/840 = 0.38.$$