COS 341, January 8, 2002

Final Exam

This Exam is open-book, open-notes, but no collaboration. **Do all three Problems.**

You should hand in your solutions during the time period January 14 - January 17, 2002 (by 11:00 am January 17) to Ms. Mitra Kelly (mkelly@cs.princeton.edu) in Room 323 of the Computer Science Department. Pick any 24-hour period during January 8 - January 17 to work on your exam.

If you have questions about the exam, or encounter difficulties related to the exam, seek help from the teaching assistants (carlk@cs.princeton.edu, jfong@cs.princeton.edu). Keep in mind that they are not allowed to give you hints on how to solve the exam problems.

You may use any results and exercises in the textbook without proof (but you should cite the page numbers where they appear in the book).

Problem 1 [20 points] Let $n > 0$ be any integer.
(a) [10 points] Let $f(n) = \sum_{0 \le k \le n} (-1)^k k^2$. Derive a closed-form formula for $f(n)$.
(b) [10 points] Let

$$g(n) = \sum_{k \ge 0} \frac{n^{4k}}{(2k)!}$$

Derive a closed-form formula for $g(n)$.

Remarks In (b) you may find useful the power series expansion $e^x = \sum_{k \ge 0} x^k/(k!)$.

Problem 2 [20 points] Let $k > 0$ be any integer. For any finite binary sequence $\beta = b_1 b_2 \cdots b_k$ of length k, let $\delta(\beta) = |-k + 2\sum_{1 \le i \le k} b_k|$. For example, $\delta(11101) = |-5 + 2 \cdot 4| = 3$, $\delta(00100) = |-5 + 2 \cdot 1| = 3$, $\delta(0010) = |-4 + 2 \cdot 1| = 2$. Let $n > 0$ be any integer. A binary sequence $\alpha = a_1 a_2 \cdots a_n \in \{0,1\}^n$ is said to be *balanced* if, for every $1 \le k \le n$, $\delta(a_1 a_2 \cdots a_k) \le 1$. For example, the sequence 100110100 is balanced, while the sequence 010011001 is not balanced as $\delta(0100) = 2 > 1$. Let c_n be the number of all balanced binary sequences of length n.

(a) [3 points] Show that $\delta(\beta)$ is equal to the absolute value of the difference of the numbers of 1's and 0's in the binary sequence β.
(b) [7 points] Derive a closed-form formula for c_n (as a function of n).
(c) [10 points] Call a binary sequence $\alpha = a_1 a_2 \cdots a_n \in \{0,1\}^n$ *almost-balanced* if, for every $1 \le k \le n$, $\delta(a_1 a_2 \cdots a_k) \le 2$. Let d_n be the number of all almost-balanced binary sequences of length n. Derive a closed-form formula for d_n.

Problem 3 [20 points] For any graph G and integer $k > 0$, let $P_G(k)$ be the number of different

k-vertex-colorings of the graph G. Let $m \geq 3$ be any integer, and let $n \geq 6$ be any even integer.

(a) [5 points] Derive a closed-form formula for $P_{F_m}(k)$, where F_m is the graph with vertex set $\{z, x_1, \cdots, x_{m-1}, y_1, y_2, \cdots, y_{m-1}\}$ and edge set

$$\{\ \{x_i, x_{i+1}\} \mid 1 \leq i \leq m-2\ \} \cup \{\ \{y_j, y_{j+1}\} \mid 1 \leq j \leq m-2\ \} \cup \{\{z, x_1\}, \{z, x_{m-1}\}, \{z, y_1\}, \{z, y_{m-1}\}\}.$$

(b) [5 points] Derive a closed-form formula for $P_{G_n}(k)$, where G_n is the graph with vertex set $\{x_0, x_1, \cdots, x_{n-1}\}$ and edge set

$$\{\ \{x_i, x_{i+1}\} \mid 0 \leq i \leq n-2\ \} \cup \{\{x_{n-1}, x_0\}, \{x_0, x_{n/2}\}\}.$$

(c) [10 points] Derive a closed-form formula for $P_{H_n}(k)$, where H_n is the graph with vertex set $\{x_0, x_1, \cdots, x_{2n-1}\}$ and edge set

$$\{\ \{x_i, x_{i+1}\} \mid 0 \leq i \leq n-2\ \} \cup \{\ \{x_{n+j}, x_{n+j+1}\} \mid 0 \leq j \leq n-2\ \} \cup \{\{x_{n-1}, x_0\}, \{x_0, x_n\}, \{x_{2n-1}, x_{n/2}\}\}.$$

Remarks F_m has $2m-1$ vertices and $2m$ edges. One can think of F_m as two m-cycles sharing exactly one common vertex. G_n has n vertices and $n+1$ edges. One can describe G_n as an n-cycle with one additional edge e that connects the vertices x_0 and $x_{n/2}$ (which are opposite each other on the cycle). H_n has $2n$ vertices and $2n+1$ edges. One can obtain H_n from G_n by placing n new vertices on the edge e (ie, replacing e with a chain of $n+1$ new edges).