Poly-HO!

polymorphic,
higher-order
programming

COS 326

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Some Design & Coding Rules

- Save some software-engineering effort:
  Never write the same code twice.

"Ooh, I get it! I'll write the code once, copy-paste it somewhere else . . . that way, I didn't write the same code twice"

  – What's wrong with that?
    - find and fix a bug in one copy, have to fix in all of them.
    - decide to change the functionality, have to track down all of the places where it gets used.

- Instead, a better practice:
  – factor out the common bits into a reusable procedure.
  – even better: use someone else's (well-tested, well-documented, and well-maintained) procedure.

Factoring Code in OCaml

Consider these definitions:

```
let rec inc_all (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (hd+1)::(inc_all tl)
```

```
let rec square_all (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (hd*hd)::(square_all tl)
```

Factoring Code in OCaml

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```
let rec square_all (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (hd*hd)::(square_all tl)
```

The code is almost identical – factor it out!

Factoring Code in OCaml

A *higher-order* function captures the recursion pattern:

```
let rec map (f:int->int) (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)
```

Factoring Code in OCaml

A *higher-order* function captures the recursion pattern:

```
let rec map (f:int->int) (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)
```

Uses of the function:

```
let inc x = x+1
let inc_all xs = map inc xs
```

Factoring Code in OCaml

A *higher-order* function captures the recursion pattern:

```
let rec map (f:int->int) (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)
```

Uses of the function:

```
let inc x = x+1
let inc_all xs = map inc xs

let square y = y*y
let square_all xs = map square xs
```

Writing little functions like inc just so we call map is a pain.

Factoring Code in OCaml

A higher-order function captures the recursion pattern:

```
let rec map (f:int->int) (xs:int list) : int list =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl);;
```

Uses of the function:

We can use an *anonymous* function instead.

Originally, Alonzo Church wrote this function using λ instead of **fun**: (λx. x+1) or (λx. x*x)

```
let inc_all xs = map (fun x -> x + 1) xs

let square_all xs = map (fun y -> y * y) xs
```

Another example

```
let rec sum (xs:int list) : int =
  match xs with
  | [] -> 0
  | hd::tl -> hd + (sum tl)


let rec prod (xs:int list) : int =
  match xs with
  | [] -> 1
  | hd::tl -> hd * (prod tl)
```

Goal: Create a function called reduce that when supplied with a few arguments can implement both sum and prod. Define sum2 and prod2 using reduce.

(Try it)

Goal: If you finish early, use map and reduce together to find the sum of the squares of the elements of a list.

(Try it)

Another example

```
let rec sum (xs:int list) : int =
  match xs with
   | [] -> b
   | hd::tl -> hd + (sum tl)


let rec prod (xs:int list) : int =
  match xs with
   | [] -> b
   | hd::tl -> hd * (prod tl)
```

Another example

```
let rec sum (xs:int list) : int =
  match xs with
  | [] -> b
  | hd::tl -> hd OP (RECURSIVE CALL ON tl)


let rec prod (xs:int list) : int =
  match xs with
  | [] -> b
  | hd::tl -> hd OP (RECURSIVE CALL ON tl)
```

Another example

```
let rec sum (xs:int list) : int =
  match xs with
  | [] -> b
  | hd::tl -> f hd (RECURSIVE CALL ON tl)


let rec prod (xs:int list) : int =
  match xs with
  | [] -> b
  | hd::tl -> f hd (RECURSIVE CALL ON tl)
```

A generic reducer

```
let add x y = x + y
let mul x y = x * y

let rec reduce (f:int->int->int) (b:int) (xs:int list) : int =
  match xs with
  | [] -> b
  | hd::tl -> f hd (reduce f b tl)

let sum xs = reduce add 0 xs
let prod xs = reduce mul 1 xs
```

Using Anonymous Functions

```
let rec reduce (f:int->int->int) (b:int) (xs:int list) : int =
  match xs with
  | [] -> b
  | hd::tl -> f hd (reduce f b tl)

let sum xs = reduce (fun x y -> x+y) 0 xs
let prod xs = reduce (fun x y -> x*y) 1 xs
```

Using Anonymous Functions

```
let rec reduce (f:int->int->int) (b:int) (xs:int list) : int =
  match xs with
   | [] -> b
   | hd::tl -> f hd (reduce f b tl)

let sum xs = reduce (fun x y -> x+y) 0 xs
let prod xs = reduce (fun x y -> x*y) 1 xs

let sum_of_squares xs = sum (map (fun x -> x * x) xs)
let pairify xs = map (fun x -> (x,x)) xs
```

Using Anonymous Functions

```
let rec reduce (f:int->int->int) (b:int) (xs:int list) : int =
  match xs with
   | [] -> b
   | hd::tl -> f hd (reduce f b tl)

let sum xs = reduce (+) 0 xs
let prod xs = reduce ( * ) 1 xs

let sum_of_squares xs = sum (map (fun x -> x * x) xs)
let pairify xs = map (fun x -> (x,x)) xs
```

Using Anonymous Functions

```
let rec reduce (f:int->int->int) (b:int) (xs:int list) : int =
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let prod xs = reduce (*) 1 xs

let sum_of_squares xs = sum (map (fun x -> x * x) xs)
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```

wrong

Using Anonymous Functions

```
let rec reduce (f:int->int->int) (b:int) (xs:int list) : int =
  match xs with
   | [] -> b
   | hd::tl -> f hd (reduce f b tl)

let sum xs = reduce (+) 0 xs
let prod xs = reduce (*) 1 xs

let sum_of_squares xs = sum (map (fun x -> x * x) xs)
let pairify xs = map (fun x -> (x,x)) xs
```

wrong -- creates a comment! ug. OCaml -0.1

what does work is: (*)

More on Anonymous Functions

Function declarations:

```
let square x = x*x

let add x y = x+y
```

are *syntactic sugar* for:

```
let square = (fun x -> x*x)

let add = (fun x y -> x+y)
```

In other words, *functions are values* we can bind to a variable, just like 3 or “moo” or true.

Functions are 2^{nd} class no more!

One argument, one result

Simplifying further:

```
let add = (fun x y -> x+y)
```

is shorthand for:

```
let add = (fun x -> (fun y -> x+y))
```

That is, add is a function which:

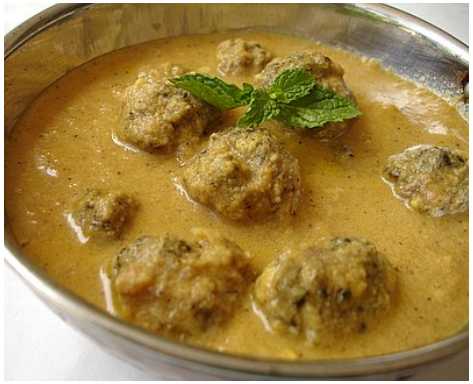
- when given a value x, *returns a function* (fun y -> x+y) which:
  - when given a value y, returns x+y.

Curried Functions

curry: *verb*

(1) to prepare or flavor with hot-tasting spices

(2) to encode a multi-argument function using nested, higher-order functions.

(1) 

(2)

```
fun x -> (fun y -> x+y) (* curried *)
fun x y -> x + y        (* curried *)
fun (x,y) -> x+y        (* uncurried *)
```

Curried Functions

Named after the logician Haskell B. Curry (1950s).

- was trying to find minimal logics that are powerful enough to encode traditional logics.
- much easier to prove something about a logic with 3 connectives than one with 20.
- the ideas translate directly to math (set & category theory) as well as to computer science.
- Actually, Moses Schönfinkel did some of this in 1924
  - thankfully, we don't have to talk about *Schönfinkelled* functions

Curry

Schönfinkel

What's so good about Currying?

In addition to simplifying the language, currying functions so that they only take one argument leads to two major wins:

1. We can *partially apply* a function.
2. We can more easily *compose* functions.

Partial Application

```
let add = (fun x -> (fun y -> x+y))
```

Curried functions allow defs of new, *partially applied* functions:

```
let inc = add 1
```

Equivalent to writing:

```
 let inc = (fun y -> 1+y)
```

which is equivalent to writing:

```
 let inc y = 1+y
```

also:

```
let inc2 = add 2
let inc3 = add 3
```

SIMPLE REASONING ABOUT HIGHER-ORDER FUNCTIONS

Reasoning About Definitions

We can factor this program

```
let square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(square_all tl)
```

into this program:

```
let square_all = map square
```

assuming we already have a definition of map

Reasoning About Definitions

```
let square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(square_all tl)
```

⇓

```
let square_all = map square
```

Goal: Rewrite definitions so my program is simpler, easier to understand, more concise, …

Question: What are the reasoning principles for rewriting programs without breaking them? For reasoning about the behavior of programs? About the equivalence of two programs?

I want some *rules* that never fail.

Simple Equational Reasoning

Rewrite 1 (Function de-sugaring):

```
let f x = body
```

==

```
let f = (fun x -> body)
```

Rewrite 2 (Substitution):

if **arg** is a value or, when executed, **will always terminate without effect** and produce a value

```
(fun x -> ... x ...) arg
```

==

```
... arg ...
```

roughly: all occurrences of x replaced by arg (though getting this *exactly* right is shockingly difficult)

Rewrite 3 (Eta-expansion):

```
let f = def
```

==

```
let f x = (def) x
```

if f has a function type

chose name x wisely so it does not shadow other names used in def

Using the rules

```
let square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(square_all tl)
```

```
let square_all = map square
```

Let's use these rules

to prove that these two functions are equivalent

Eliminating the Sugar in Map

```
let rec map f xs =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)
```

Eliminating the Sugar in Map

```
let rec map f xs =
  match xs with
  | [] -> []
  | hd::tl -> (f hd)::(map f tl)

let rec map =
  (fun f ->
    (fun xs ->
       match xs with
       | [] -> []
       | hd::tl -> (f hd)::(map f tl)))
```

Consider square_all

```
let rec map =
  (fun f ->
    (fun xs ->
        match xs with
        | [] -> []
        | hd::tl -> (f hd)::(map f tl)))

let square_all =
   map square
```

Substitute map definition into square_all

```
let rec map =
  (fun f ->
    (fun xs ->
       match xs with
       | [] -> []
       | hd::tl -> (f hd)::(map f tl)))

let square_all =
   (fun f ->
       (fun xs ->
          match xs with
          | [] -> []
          | hd::tl -> (f hd)::(map f tl)
       )
   ) square
```

Substitute map definition into square_all

```
let rec map =
  (fun f ->
    (fun xs ->
       match xs with
       | [] -> []
       | hd::tl -> (f hd)::(map f tl)))

let square_all =
   (fun f ->
       (fun xs ->
          match xs with
          | [] -> []
          | hd::tl -> (f hd)::(map f tl)
       )
   ) square
```

Substitute map definition into square_all

```
let rec map =
  (fun f ->
    (fun xs ->
       match xs with
       | [] -> []
       | hd::tl -> (f hd)::(map f tl)))

let square_all =
   (fun f ->
      (fun xs ->
         match xs with
         | [] -> []
         | hd::tl -> (f hd)::(map f tl)
      )
   ) square
```

Substitute Square

```
let rec map =
  (fun f ->
    (fun xs ->
       match xs with
       | [] -> []
       | hd::tl -> (f hd)::(map f tl)))

let square_all =
   (
     (fun xs ->
        match xs with
        | [] -> []
        | hd::tl -> (square hd)::(map square tl)
     )
```

argument square substituted for parameter f

Expanding map square

```
let rec map =
   (fun f ->
      (fun xs ->
           match xs with
           | [] -> []
           | hd::tl -> (f hd)::(map f tl)))

let square_all ys =

          (fun xs ->
               match xs with
               | [] -> []
               | hd::tl -> (square hd)::(map square tl)
          ) ys
```

add argument
via eta-expansion

Expanding map square

```
let rec map =
   (fun f ->
      (fun xs ->
         match xs with
         | [] -> []
         | hd::tl -> (f hd)::(map f tl)))

let square_all ys =
```

substitute again (argument ys for parameter xs)

```
            match ys with
            | [] -> []
            | hd::tl -> (square hd)::(map square tl)
```

So Far

```
let rec map f xs =
      match xs with
      | [] -> []
      | hd::tl -> (f hd)::(map f tl)

let square_all xs = map square xs
```

```
let square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(map square tl)
```

proof by simple rewriting unrolls definition once

Next Step

```
let rec map f xs =
      match xs with
      | [] -> []
      | hd::tl -> (f hd)::(map f tl)

let square_all xs = map square xs
```

proof by simple rewriting unrolls definition once

```
let square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(map square tl)
```

proof *by induction* eliminates recursive function map

```
let rec square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(square_all tl)
```

Summary

We saw this:

```
let rec map f xs =
      match xs with
      | [] -> []
      | hd::tl -> (f hd)::(map f tl);;

let square_all = map square
```

Is equivalent to this:

```
let square_all ys =
  match ys with
    | [] -> []
    | hd::tl -> (square hd)::(map square tl)
```

Morals of the story:

(1) OCaml's *HOT* (higher-order, typed) functions capture recursion patterns

(2) we can figure out what is going on by *equational reasoning*.

(3) ... but we typically need to do *proofs by induction* to reason about recursive (inductive) functions