Exercises on Expectation

COS 302, Fall 2020

PRINCETON UNIVERSITY

Exercise #1

Let $f(x) = x^2 + 2$, and let X be a uniform random variable on $[-1, 1]$
What is the expected value of $f(x)$?

$$
\begin{aligned}
\mathbb{E}_X\left[f(x)\right] &= \int f(x)\,p(x)\,dx \\
&= \int_{-1}^{1} (x^2 + 2)\,\tfrac{1}{2}\,dx \\
&= \tfrac{1}{6}x^3 + x\,\Big|_{x=-1}^{1} \\
&= \tfrac{1}{6} + 1 + \tfrac{1}{6} + 1 \\
&= 2^1\!/_3 = 2.333
\end{aligned}
$$

Exercise #1

Let $f(x) = x^2 + 2$, and let X be a uniform random variable on $[-1, 1]$
What is the variance of $f(x)$?

$$
\begin{aligned}
\mathbb{E}_X\left[(f(x) - \mathbb{E}_X[f(x)])^2\right] &= \int \left(f(x) - 2\tfrac{1}{3}\right)^2 p(x)\,dx \\
&= \int_{-1}^{1} (x^2 - \tfrac{1}{3})^2 \, \tfrac{1}{2}\,dx \\
&= \tfrac{1}{10}x^5 - \tfrac{1}{9}x^3 + \tfrac{1}{18}x \,\Big|_{x=-1}^{1} \\
&= 2\left(\tfrac{1}{10} - \tfrac{1}{9} + \tfrac{1}{18}\right) \\
&= \tfrac{4}{45} = 0.0889
\end{aligned}
$$

Exercise #1

Let $f(x) = x^2 + 2$, and let X be a uniform random variable on $[-1, 1]$
What is the variance of $f(x)$?

```
import numpy as np
x = np.random.rand(1000000) * 2 - 1
fx = x**2 + 2
np.mean(fx)
        2.3331604923657459
np.var(fx)
        0.088781780725855644
```

Exercise #2

Let $f(x, y) = xy^2$, and let X and Y be i.i.d. uniform random variables on $[0, 3]$
What is the expected value of $f(x, y)$?

$$
\begin{aligned}
\mathbb{E}_{X,Y}\left[f(x,y)\right] &= \iint f(x,y)\, p(x,y)\, dx\, dy \\
&= \int_0^3\!\!\int_0^3 (xy^2)\, \tfrac{1}{9}\, dx\, dy \\
&= \tfrac{1}{54} x^2 y^3 \Big|_{x=0}^{3} \Big|_{y=0}^{3} \\
&= \tfrac{243}{54} \\
&= 4\,{}^1\!/_2 = 4.5
\end{aligned}
$$

Exercise #2

Let $f(x, y) = xy^2$, and let X and Y be i.i.d. uniform random variables on $[0, 3]$
What is the variance of $f(x, y)$?

$$
\begin{aligned}
\mathbb{E}_{X,Y}\left[\left(f(x,y) - \mathbb{E}_{X,Y}\left[f(x,y)\right]\right)^2\right] &= \iint \left(f(x,y) - 4^1\!/_2\right)^2 \; p(x,y)\, dx\, dy \\
&= \int_0^3\!\!\int_0^3 (xy^2 - 4^1\!/_2)^2 \tfrac{1}{9}\, dx\, dy \\
&= \tfrac{1}{135}x^3y^5 - \tfrac{1}{6}x^2y^3 + \tfrac{9}{4}xy \,\Big|_{x=0}^{3}\,\Big|_{y=0}^{3} \\
&= \frac{243}{5} - \frac{81}{2} + \frac{81}{4} \\
&= \frac{567}{20} = 28.35
\end{aligned}
$$

Exercise #2

Let $f(x, y) = xy^2$, and let X and Y be i.i.d. uniform random variables on $[0, 3]$
What is the variance of $f(x, y)$?

```
import numpy as np
x = np.random.rand(1000000) * 3
y = np.random.rand(1000000) * 3
fxy = x * y**2
np.mean(fxy)
        4.493337502402337
np.var(fxy)
        28.337876713469225
```

Exercise #3

Let $f(\boldsymbol{x}) = \begin{bmatrix} 2x_1 \\ x_1 + x_2 \end{bmatrix}$, and let X_1 and X_2 be i.i.d. uniform random variables on $[0, 1]$

What is the expected value of $f(\boldsymbol{x})$?

$$
\begin{aligned}
\mathbb{E}_{X_1,X_2}[f(\boldsymbol{x})] &= \iint f(\boldsymbol{x})\, p(\boldsymbol{x})\, dx_1\, dx_2 \\
&= \begin{bmatrix} \int_0^1\int_0^1 2x_1\, 1\, dx_1\, dx_2 \\ \int_0^1\int_0^1 (x_1 + x_2)\, 1\, dx_1\, dx_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ \frac{1}{2} + \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

Exercise #3

Let $f(\boldsymbol{x}) = \begin{bmatrix} 2x_1 \\ x_1 + x_2 \end{bmatrix}$, and let X_1 and X_2 be i.i.d. uniform random variables on $[0, 1]$

What is the variance (i.e., covariance matrix) of $f(\boldsymbol{x})$?

$$\mathbb{E}_{X_1,X_2}\left[\left(\boldsymbol{x} - \mathbb{E}_{X_1,X_2}\left[f(\boldsymbol{x})\right]\right)\left(\boldsymbol{x} - \mathbb{E}_{X_1,X_2}\left[f(\boldsymbol{x})\right]\right)^\top\right]$$

$$= \iint \begin{bmatrix} 2x_1 - 1 \\ x_1 + x_2 - 1 \end{bmatrix} \begin{bmatrix} 2x_1 - 1 \\ x_1 + x_2 - 1 \end{bmatrix}^\top 1 \, dx_1 \, dx_2$$

$$= \iint \begin{bmatrix} (2x_1 - 1)^2 & (2x_1 - 1)(x_1 + x_2 - 1) \\ (2x_1 - 1)(x_1 + x_2 - 1) & (x_1 + x_2 - 1)^2 \end{bmatrix} dx_1 \, dx_2$$

$$= \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/6 \end{bmatrix}$$

Exercise #3

Let $f(\boldsymbol{x}) = \begin{bmatrix} 2x_1 \\ x_1 + x_2 \end{bmatrix}$, and let X_1 and X_2 be i.i.d. uniform random variables on $[0, 1]$

What is the variance (i.e., covariance matrix) of $f(\boldsymbol{x})$?

```
import numpy as np
x1 = np.random.rand(1000000)
x2 = np.random.rand(1000000)
fx = np.stack((2 * x1, x1 + x2))
np.mean(fx, axis=1)
array([ 0.99905021,  0.99993117])
np.cov(fx)
array([[ 0.33363891,  0.16690159],
       [ 0.16690159,  0.16670841]])
```